Conformal Deformation on Manifolds with Boundary

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Abstract

We consider natural conformal invariants arising from the Gauss-Bonnet formulas on manifolds with boundary, and study conformal deformation problems associated to them.

The purpose of this paper is to study conformal deformation problems associated to conformal invariants on manifolds with boundary. From analysis point of view, the problem becomes a non-Dirichlet boundary value problems for fully nonlinear equations. This may be compared to a work by Lieberman-Trudinger [22] on the oblique-type boundary value problems.

Let (M, g) be a compact, connected Riemannian manifold of dimension $n \geq 3$ with boundary ∂M. We denote the Riemannian curvature, Ricci curvature, scalar curvature, mean curvature, and the second fundamental form by $Riem, Ric, R, h$, and $L_{\alpha\beta}$, respectively.

The Yamabe constant for compact manifolds with boundary is a conformal invariant, defined as

$$Y(M, \partial M, [g]) = \inf_{\hat{g} \in [g], V_{\hat{g}}=1} (\int_M R_{\hat{g}} + \oint_{\partial M} h_{\hat{g}}),$$

where $[g]$ is the conformal class of g. It was proved by Escobar [9] that for most compact manifolds with boundary, the Yamabe problem is solvable; i.e., there exists a conformal metric such that the scalar curvature is constant and the mean curvature is zero.

To study a nonlinear version of the Yamabe problem, we consider the Schouten tensor defined as

$$A_g = \frac{1}{n-2}(Ric - \frac{R}{2(n-1)}g).$$

The problem consists in finding a metric $\hat{g} = e^{-2u}g$ such that the $\sigma_k(A_{\hat{g}})$ curvature is constant, where σ_k is the kth elementary symmetric function of the eigenvalues of $A_{\hat{g}}$. When $k = 1$, the problem reduces to the original Yamabe problem.

In dimension four, the $\sigma_2(A_g)$ curvature is related to the Gauss-Bonnet formula and $\int_M \sigma_2(A_g)$ is a conformal invariant on closed manifolds. Chang-Gursky-Yang [3], [4] proved

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that if the Yamabe constant $Y(M,[g])$ and $\int_M \sigma_2(A_g)$ are both positive, then we can find a conformal metric $\hat{g}$ such that $\sigma_2(A_{\hat{g}})$ is a positive constant; see also [18]. For locally conformally flat closed manifolds, Li-Li [21] and Guan-Wang [16] proved that if $\sigma_i(A_g) > 0$, for $1 \leq i \leq k$, then we can find a conformal metric $\hat{g}$ such that $\sigma_k(A_{\hat{g}})$ is constant. When $2k > n$, the result was generalized by Gursky-Viaclovsky [17] to non locally conformally flat closed manifolds; see also Trudinger-Wang [27]. Other related works include Guan-Lin-Wang [15], Ge-Wang [12] and Sheng-Trudinger-Wang [26].

Let M be a four-manifold with boundary. The Gauss-Bonnet formula is

$$32\pi^2\chi(M,\partial M) = \int_M |\mathcal{W}|^2 + 16\left(\int_M \sigma_2(A_g) + \frac{1}{2}\oint_{\partial M} \mathcal{B}_g\right), \tag{1}$$

where $\mathcal{B}_g = \frac{1}{2}Rh - R_{nn}h - R_{\gamma\alpha\gamma\beta}L^{\alpha\beta} + \frac{1}{3}h^3 - h|L|^2 + \frac{2}{3}trL^3$, and $\int_M \sigma_2 + \frac{1}{2}\oint_{\partial M}\mathcal{B}_g$ is a conformal invariant. We have the following existence result. Recall that the boundary ∂M is called umbilic if $L_{\alpha\beta} = \mu(x)g_{\alpha\beta}$, which is a conformal invariant condition.

Theorem 1. *Let (M,g) be a compact connected four-manifold with umbilic boundary. If $Y(M,\partial M,[g])$ and $\int_M \sigma_2(A_g) + \frac{1}{2}\oint_{\partial M}\mathcal{B}_g$ are both positive, then there exists a metric $\hat{g} \in [g]$ such that $\sigma_2(A_{\hat{g}})$ is a positive constant and $\mathcal{B}_{\hat{g}}$ is zero.*

We will prove a more general result than Theorem 1.

Theorem 2. *Let (M,g) be a compact connected four-manifold with umbilic boundary. Suppose that (M,g) is not conformally equivalent to $(\mathbb{S}^+_4, g_c)$, where g_c is the standard metric on the hemisphere. If $Y(M,\partial M,[g])$ and $\int_M \sigma_2 + \frac{1}{2}\oint_{\partial M}\mathcal{B}_g$ are both positive, then given a positive function f, there exists a metric $\hat{g} \in [g]$ such that $\sigma_2(A_{\hat{g}}) = f$ and $\mathcal{B}_{\hat{g}}$ is zero.*

An application of above theorem to Einstein manifolds is given in Section 2.3.

For general k, we define suitable boundary curvatures and show variational properties of σ_k. Let $A^T = [A_{\alpha\beta}]$ be the tangential part of the Schouten tensor. Define

$$\mathcal{B}^2 = \begin{cases} \frac{2}{n-2}\sigma_{2,1}(A^T,L) + \frac{2}{(n-2)(n-3)}\sigma_{3,0}(A^T,L) & n \geq 4 \\ 2\sigma_{2,1}(A^T,L) + \frac{1}{3}h^3 - \frac{1}{2}h|L|^2 & n = 3, \end{cases} \tag{2}$$

where $\sigma_{i,j}$'s are the mixed symmetric functions; see Section 1. For $k \geq 3$, we define

$$\mathcal{B}^k = \sum_{i=0}^{k-1} C_1(n,k,i)\sigma_{2k-i-1,i}(A^T,L) \quad n \geq 2k, \tag{3}$$

where $C_1(n,k,i) = \frac{(2k-i-1)!(n-2k+i)!}{(n-k)!(2k-2i-1)!!\,i!}$ and !! stands for the double factorial. When the boundary is umbilic, we define

$$\mathcal{B}^k = \sum_{i=0}^{k-1} C_2(n,k,i)\sigma_i(A^T)\mu^{2k-2i-1} \tag{4}$$

for all n, where $C_2(n,k,i) = \frac{(n-i-1)!}{(n-k)!(2k-2i-1)!!}$. In Section 1, we will show that the above two definitions of $\mathcal{B}^k$ coincide when the boundary is umbilic.

Let $\mathcal{F}_k(g) = \int_M \sigma_k(A) + \oint_{\partial M}\mathcal{B}^k$ and $\mathcal{M} = \{g : g \in [g_0], V_g = 1\}$.

Theorem 3. *Let (M, g_0) be a compact manifold of dimension $n \geq 3$ with boundary.*
(a) Suppose $n \neq 4$. Then g is a critical point of $\mathcal{F}_2 \mid_{\mathcal{M}}$ if and only if g satisfies $\sigma_2(A_g) = constant$ in M with $\mathcal{B}_g^2 = 0$ on ∂M.
(b) Suppose $n > 2k$ and M is a locally conformally flat compact manifold. Then g is a critical point of $\mathcal{F}_k \mid_{\mathcal{M}}$ if and only if g satisfies $\sigma_k(A_g) = constant$ in M with $\mathcal{B}_g^k = 0$ on ∂M.
(c) The statement of (b) is true for all $n \neq 2k$ if we assume in addition that the boundary is umbilic.

If we add local conformal invariants to $\mathcal{B}^k$, similarly we have:

Corollary 1. *Suppose $\mathcal{L}$ is a curvature tensor on ∂M satisfying $\mathcal{L}(\hat{g}) = e^{(2k-1)u}\mathcal{L}(g)$. Then under the same conditions as in Theorem 3, g is a critical point of $(\mathcal{F}_k + \oint \mathcal{L}) \mid_{\mathcal{M}}$ if and only if g satisfies $\sigma_k(A_g) = constant$ in M with $\mathcal{B}_g^k + \mathcal{L} = 0$ on ∂M.*

For closed manifolds, Theorem 3 was proved by Viaclovsky [28]. He also showed that $\mathcal{F}_{\frac{n}{2}}$ is a conformal invariant associated to the Gauss-Bonnet formula. We will show a generalization of this fact for manifolds with boundary in Section 4 (Proposition 3).

We study the problem of finding a conformal metric $\hat{g}$ such that $\sigma_k(A_{\hat{g}})$ is constant and $\mathcal{B}_{\hat{g}}^k = 0$. For $k = 2$ and $n = 4$, Theorem 1 shows that the problem is solvable under some conformal invariant conditions because when the boundary is umbilic, $\mathcal{B} = 2\mathcal{B}^2$. We remark that the boundary condition we find here in general involves second derivatives, which is highly nonlinear. Such boundary condition is rare in the literature.

We introduce some definitions and then state the result for general k. Let W be a matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$. For $k \leq n$, $\sigma_k(W) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}$ is called the kth elementary symmetric function of the eigenvalues of W. The set $\Gamma_k^+ = \{\lambda : \sigma_i(\lambda) > 0, 1 \leq i \leq k\}$ is called the positive k-cone, which is is an open convex cone with vertex at the origin [11]. Now we can define higher order Yamabe constants for manifolds with boundary. When $\{\hat{g} : \hat{g} \in [g], A_{\hat{g}} \in \Gamma_{k-1}^+\}$ is nonempty, let $\mathcal{Y}_k = \inf \mathcal{F}_k(\hat{g})$ where inf is taken over metrics $g \in [g]$ with $A_{\hat{g}} \in \Gamma_{k-1}^+$ and $V_{\hat{g}} = 1$. When $\{\hat{g} : \hat{g} \in [g], A_{\hat{g}} \in \Gamma_{k-1}^+\} = \emptyset$, let $\mathcal{Y}_k = -\infty$. We denote $\mathcal{Y}_1 = Y(M, \partial M, [g])$. For closed manifolds, $\mathcal{Y}_k$ was defined by Guan-Lin-Wang [15]. For locally conformally flat closed manifolds, Guan-Lin-Wang [15] proved that if $\mathcal{Y}_k > 0$ and $2k \leq n$, there exists $\hat{g} \in [g]$ such that $\sigma_k(A_{\hat{g}}) = 1$. For manifolds with boundary, we have:

Theorem 4. *Let (M, g) be a locally conformally flat compact manifold of dimension $n \geq 3$ with umbilic boundary. Suppose that $2k \leq n$ and $\mathcal{Y}_1, \cdots, \mathcal{Y}_k > 0$. Then there exists a metric $\hat{g} \in [g]$ such that $\sigma_k(A_{\hat{g}}) = 1$ and $\mathcal{B}_{\hat{g}}^k = 0$.*

Proofs of Theorem 1, 2 and 4 turn out by solving some boundary value problems for fully nonlinear equations. Under the conformal change of the metric $\hat{g} = e^{-2u}g$, the Schouten tensor $\hat{A}$ satisfies

$$\hat{A} = \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g. \tag{5}$$

The second fundamental form satisfies $\hat{L}e^u = \frac{\partial u}{\partial n} g + L_g$, where n is the unit inner normal. When the boundary is umbilic, the formula becomes $\hat{\mu} e^{-u} = \frac{\partial u}{\partial n} + \mu_g$. We will show in Section 1 that when $A_g \in \Gamma_k^+$ and when the boundary is umbilic, then $\mathcal{B}_g^k = 0$ if and only if $h_g = 0$. Thus, the problem becomes solving

$$\begin{cases} \sigma_k^{\frac{1}{k}}(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g) = e^{-2u} & in\, M \\ \frac{\partial u}{\partial n} + \mu_g = 0 & on\, \partial M. \end{cases} \tag{6}$$

We will prove boundary estimates for equations more general than (6). We use Fermi coordinates in a boundary neighborhood. Define the half ball by $\overline{B}_r^+ = \{x_n \geq 0, \sum_i x_i^2 \leq r^2\}$ and the segment on the boundary by $\Sigma_r = \{x_n = 0, \sum_i x_i^2 \leq r^2\}$. Let $f(x,z) : M^n \times \mathbb{R} \to \mathbb{R}^+$. Consider the equation

$$\begin{cases} F(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + S(x)) = f(x,u) & in\, \overline{B}_r^+ \\ \frac{\partial u}{\partial n} + \mu_g = \hat{\mu}\, e^{-u} & on\, \Sigma_r, \end{cases} \tag{7}$$

where F satisfies some structure conditions as we describe now. Let Γ be an open convex cone in $\mathbb{R}^n$ with vertex at the origin satisfying $\Gamma_n^+ \subset \Gamma \subset \Gamma_1^+$. Suppose that $F(\lambda) = F(\sigma_1(\lambda), \cdots, \sigma_n(\lambda)) \in C^\infty(\Gamma) \cap C^0(\overline{\Gamma})$ is a homogeneous symmetric function of degree one normalized with $F(e) = F(1, \cdots, 1) = 1$. Assume that $F = 0$ on $\partial\Gamma$ and F satisfies the following in Γ :

(S0) F is positive;
(S1) F is concave (i.e., $\frac{\partial^2 F}{\partial\lambda_i \partial\lambda_j}$ is negative semi-definite);
(S2) F is monotone (i.e., $\frac{\partial F}{\partial \lambda_i}$ is positive);
(S3) $\frac{\partial F}{\partial \lambda_i} \geq \epsilon \frac{F}{\sigma_1}$, for some constant $\epsilon > 0$, for all i.

In some case, we need an additional condition:
(A) $\sum_{j \neq i} \frac{\partial F}{\partial \lambda_j} \leq \rho \frac{\partial F}{\partial \lambda_i}$, for some $\rho > 0$, for all $\lambda \in \Gamma$ with $\lambda_i \leq 0$.

It was shown in [7] that $\binom{n}{k}^{-\frac{1}{k}} \sigma_k^{\frac{1}{k}}$ satisfies the structure conditions (S0)-(S3) and (A) in Γ_k^+ with $\epsilon = \frac{1}{k}$ and $\rho = (n-k)$.

We assume that $S(x)$ satisfies the following conditions on the boundary:
(T0) $S_{\alpha n} = \mu_\alpha$;
(T1) $S_{\alpha\beta} + S_{nn} g_{\alpha\beta} \leq R_{\alpha n \beta n}$;
(T2) $S_{\alpha\beta,n} - 2\mu S_{\alpha\beta} \leq \mu_{\tilde{\alpha}\tilde{\beta}} - R_{\alpha n \beta n}\mu$,

where $\mu_{\tilde{\alpha}\tilde{\beta}}$ means covariant derivatives of μ with respect to the induced metric $g_{\alpha\beta}$ on the boundary.

Denote
$c_{inf}(r) = \inf_{x \in \overline{B}_r^+} f(x,u)$;
$c_{sup}(r) = \sup_{x \in \overline{B}_r^+} (f + |\nabla_x f(x,u)| + |f_z(x,u)| + |\nabla_x^2 f(x,u)| + |\nabla_x f_z(x,u)| + |f_{zz}(x,u)|)$.

Theorem 5. *Let F satisfy (S0)-(S3) in a corresponding cone Γ and $S(x)$ satisfy (T0)-(T2) on Σ_r. Suppose that $|\nabla_x f| \leq \Lambda f$ and $|f_z| \leq \Lambda f$ for some number Λ, and Σ_r is*

umbilic with principal curvatures μ. Suppose $u \in C^4$ is a solution to the equation (7).
Case(a). If $\hat{\mu} = 0$, then

$$\sup_{x \in \overline{B}^+_{\frac{r}{2}}} (|\nabla u|^2 + |\nabla^2 u|) \leq C,$$

where $C = C(r, n, \epsilon, \mu, \Lambda, \|S\|_{C^2(\overline{B}^+_r)}, \|g\|_{C^3}, c_{sup}(r))$
Case(b). Suppose that F satisfies the additional condition (A) and $\Gamma_2^+ \subset \Gamma$. If $\hat{\mu}$ is a positive constant, then

$$\sup_{x \in \overline{B}^+_{\frac{r}{2}}} (|\nabla u|^2 + |\nabla^2 u|) \leq C,$$

where $C = C(r, n, \epsilon, \rho, \mu, \hat{\mu}, \Lambda, \|S\|_{C^2(\overline{B}^+_r)}, \|g\|_{C^3}, \inf_{\overline{B}^+_r} u, c_{sup}(r))$.

When the manifolds are locally conformally flat on the boundary, we will show in Section 1 that A_g satisfies the conditions (T0)-(T2). Denote the Weyl tensor by $\mathcal{W}_{ijkl}$ and the Cotten tensor by $\mathcal{C}_{ijk} = A_{ij,k} - A_{ik,j}$. Then we have the following Corollary.

Corollary 2. *Let F satisfy (S0)-(S3) in a corresponding cone Γ. Suppose that Σ_r is umbilic with principal curvatures μ and n is the unit inner normal with respect to g. Suppose $\mathcal{W}_{ijkl} = 0$ and $\mathcal{C}_{ijk} = 0$ on Σ_r. Let $u \in C^4$ be a solution to the equation*

$$\begin{cases} F(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g) = f(x)e^{-2u} & in\, \overline{B}^+_r \\ \frac{\partial u}{\partial n} + \mu = \hat{\mu}\, e^{-u} & on\, \Sigma_r. \end{cases} \tag{8}$$

Case(a). If $\hat{\mu} = 0$, then

$$\sup_{x \in \overline{B}^+_{\frac{r}{2}}} (|\nabla u|^2 + |\nabla^2 u|) \leq C,$$

where C depends on $r, n, \epsilon, \mu, \inf_{\overline{B}^+_r} u, \|g\|_{C^4}, \|f\|_{C^2(\overline{B}^+_r)}$ and $\inf_{\overline{B}^+_r} f$.
Case(b). Suppose that F satisfies the additional condition (A) and $\Gamma_2^+ \subset \Gamma$. If $\hat{\mu}$ is a positive constant, then

$$\sup_{x \in \overline{B}^+_{\frac{r}{2}}} (|\nabla u|^2 + |\nabla^2 u|) \leq C,$$

where C depends on $r, n, \epsilon, \rho, \mu, \hat{\mu}, \inf_{\overline{B}^+_r} u, \|g\|_{C^4}, \|f\|_{C^2(\overline{B}^+_r)}$ and $\inf_{\overline{B}^+_r} f$.

The next estimates concern the σ_2 equation. Let $A^t = A + \frac{1-t}{2}(trA)g$; see [18]. Under the conformal change, the tensor $\hat{A}^t$ satisfies

$$\hat{A}^t = \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + \frac{1-t}{2}(\Delta u - \frac{n-2}{2}|\nabla u|^2)g + A^t.$$

Consider the equation

$$\begin{cases} \sigma_2^{\frac{1}{2}}(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + \frac{1-t}{2}(\Delta u - \frac{n-2}{2}|\nabla u|^2)g + A^t + S) = f(x, u) & in\, \overline{B}^+_r \\ \frac{\partial u}{\partial n} + \mu_g = 0 & on\, \Sigma_r, \end{cases} \tag{9}$$

where $S(x)$ is a $(0, 2)$-tensor and $f(x, u)$ is positive.

Theorem 6. *Let $n \geq 4$. Suppose that Σ_r is umbilic with principal curvatures μ. Let $u^t \in C^4$ be a solution to the equation (9).*

(a) When $t = 1$, we have

$$\sup_{x \in \overline{B}^+_{\frac{r}{2}}} |\nabla u|^2 \leq C_3,$$

where $C_3 = C_3(n, r, \|g\|_{C^4}, \|S\|_{C^2(\overline{B}^+_r)}, c_{\sup}(r))$ but is independent of $c_{\inf}(r)$.

(b) Let $-\Theta \leq t \leq 1$. Suppose in addition that S satisfies $S_{\alpha n} = 0$ and $g^{\alpha\beta}(S_{\alpha\beta,n} - 2\mu S_{\alpha\beta}) \leq 0$ on Σ_r. Then

$$\sup_{x \in \overline{B}^+_{\frac{r}{2}}} (|\nabla u|^2 + |\nabla^2 u|) \leq C_4,$$

where $C_4 = C_4(n, r, \Theta, \|g\|_{C^4}, \|S\|_{C^2(\overline{B}^+_r)}, c_{\sup}(r), c_{\inf}(r))$.

The main technique we use in proving Theorem 5 and 6 is to derive boundary C^2 estimates directly from boundary C^0 estimates. Such idea has appeared before in the work by Chen [6] for local C^2 estimates for a large class of equations. (See [16] for a related work.) The same idea has also been applied to boundary estimates in [7]. To control boundary behaviors, we do not construct a barrier function. Instead, we estimate the third derivatives uniformly on the boundary. Then the maximum of second derivatives must happen in the interior.

Finally, we remark that the conformal invariants condition in Theorem 1, 2 and 4 is necessary. A counterexample can be constructed on a cylinder if the condition does not hold. We also remark that the Dirichlet problem for the Schouten tensor equations was studied by Guan [14]. The Neumann problems and non-Dirichlet problems are, on the other hand, not yet well studied.

This paper is organized as follows. We start with some background in Section 1. In Sections 2, we prove Theorems 1, 2 and their application. We give proofs of Theorems 3 and Corollary 1 in Section 3. In Section 4, we prove Theorem 4 and Proposition 3. At the end, we prove boundary estimates. The proofs of Theorem 5 and Corollary 2, and Theorem 6 are in Sections 5 and 6, respectively.

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1 Background

We give some basic facts about homogeneous symmetric functions.

Lemma 1. *(see [6]). Let Γ be an open convex cone with vertex at the origin satisfying $\Gamma_n^+ \subset \Gamma$,and let $e = (1, \cdots, 1)$ be the identity. Suppose that F is a homogeneous symmetric function of degree one normalized with $F(e) = 1$, and that F is concave in Γ. Then*

(a) $\sum_i \lambda_i \frac{\partial F(\lambda)}{\partial \lambda_i} = F(\lambda), \quad$ for $\lambda \in \Gamma$;

(b) $\sum_i \frac{\partial F(\lambda)}{\partial \lambda_i} \geq F(e) = 1, \quad$ for $\lambda \in \Gamma$.

Now we list further properties of elementary symmetric functions.

Lemma 2. *(see [6]). Let $G = \sigma_k^{\frac{1}{k}}, k \leq n$. Then*

(a) G is positive and concave in Γ_k^+.

(b) G is monotone in Γ_k^+, i.e., the matrix $G^{ij} = \frac{\partial G}{\partial W_{ij}}$ is positive definite.

(c) Suppose $\lambda \in \Gamma_k^+$. For $0 \leq l < k \leq n$, the following is the Newton-MacLaurin inequality

$$k(n-l+1)\sigma_{l-1}\sigma_k \leq l(n-k+1)\sigma_l\sigma_{k-1}.$$

Let W be an $m \times m$ matrix. $T_k(W) = \sigma_k\, I - \sigma_{k-1} W + \cdots + (-1)^k W^k$ is called the kth Newton tensor of W; [25]. We have the recursive formula $T_k(W) = \sigma_k(W)\, I - T_{k-1}(W)W$. Furthermore, $\frac{\partial \sigma_k(W)}{\partial W_{ij}} = T_{k-1}^{ij}(W)$ and $tr\ T_k(W) = (m-k)\sigma_k(W)$.

We introduce some more notations. Given an $n \times n$ matrix A, denote the upper left $(n-1)\times(n-1)$ sub-matrix by $A^T = [A_{\alpha\beta}]$. The Greek letters $1 \leq \alpha, \beta, \gamma \leq n-1$ stand for the tangential indices and the letters $1 \leq i, j, k \leq n$ stand for the full indices unless otherwise noted. The Kronecker symbol $\begin{pmatrix} i_1 \cdots i_q \\ j_1 \cdots j_q \end{pmatrix}$ is defined as in [25].

Lemma 3. *Let A be an $n \times n$ matrix.*

(a) $\sigma_q(A) = \frac{1}{q!}\sum \begin{pmatrix} i_1 \cdots i_q \\ j_1 \cdots j_q \end{pmatrix} A_{i_1}^{j_1} \cdots A_{i_q}^{j_q}$;

(b) $T_q(A)_j^i = \frac{1}{q!}\sum \begin{pmatrix} i_1 \cdots i_q\, i \\ j_1 \cdots j_q\, j \end{pmatrix} A_{i_1}^{j_1} \cdots A_{i_q}^{j_q}$;

(c) $T_q(A)_n^n = \sigma_q(A^T)$;

(d) $T_q(A)_n^\alpha = -T_{q-1}(A^T)_\beta^\alpha A_n^\beta$.

Proof. For (a) and (b), see [25]. (c) is directly from (a) and (b). (d) follows by an observation that $T_q(A)_n^\alpha = \frac{1}{(q-1)!}\sum \begin{pmatrix} i_1 \cdots i_{q-1}\, n\, \alpha \\ j_1 \cdots j_{q-1}\, j_q\, n \end{pmatrix} A_{i_1}^{j_1} \cdots A_{i_{q-1}}^{j_{q-1}} A_n^{j_q}$.

□

We define the mixed symmetric functions and Newton tensors:

Definition 1. *Let A and B be $m \times m$ matrices. Then*

$$\sigma_{q,r}(A,B) = \frac{1}{q!}\sum \begin{pmatrix} i_1 \cdots i_q \\ j_1 \cdots j_q \end{pmatrix} A_{i_1}^{j_1} \cdots A_{i_r}^{j_r} B_{i_{r+1}}^{j_{r+1}} \cdots B_{i_q}^{j_q};$$

$$T_{q,r}(A,B)_j^i = \frac{1}{q!}\sum \begin{pmatrix} i_1 \cdots i_q\, i \\ j_1 \cdots j_q\, j \end{pmatrix} A_{i_1}^{j_1} \cdots A_{i_r}^{j_r} B_{i_{r+1}}^{j_{r+1}} \cdots B_{i_q}^{j_q}.$$

Denote a variation of a tensor A by A'. The next lemma is used in proving Theorem 3.

Lemma 4. *Let A and B be $m \times m$ matrices. Suppose that ${A'}_i^j = kA_i^j\phi + M_i^j$ and ${B'}_i^j = lB_i^j\phi + N_i^j$. Then*

(a) $T_{q,r}(A,B)_j^i A_i^j = (q+1)\sigma_{q+1,r+1}(A,B)$

(b) $$\sigma'_{q+1,r+1}(A,B) = (k(r+1) + l(q-r))\sigma_{q+1,r+1}(A,B)\phi + \frac{r+1}{q+1}T_{q,r}(A,B)_j^i M_i^j + \frac{q-r}{q+1}T_{q,r+1}(A,B)_j^i N_i^j$$

(c) $\sigma'_{q+1}(A) = k(q+1)\sigma_{q+1}(A)\phi + T_q(A)_j^i M_i^j$.

Proof. (a) follows by definitions; see [25], and (c) follows by (b) by letting $r = q$. For (b),

$$T'_{q,r}(A,B)^j_i = (kr + l(q-r))T_{q,r}(A,B)^i_j\phi + \tfrac{1}{q!}\sum \left(\begin{array}{c} i_1 \cdots i_q\, i \\ j_1 \cdots j_q\, j \end{array}\right) \times$$
$$\left[\sum_{k=1}^r A^{j_1}_{i_1}\cdots M^{j_k}_{i_k}\cdots A^{j_r}_{i_r}B^{j_{r+1}}_{i_{r+1}}\cdots B^{j_q}_{i_q} + \sum_{k=r+1}^q A^{j_1}_{i_1}\cdots A^{j_r}_{i_r}B^{j_{r+1}}_{i_{r+1}}\cdots N^{j_k}_{i_k}\cdots B^{j_q}_{i_q}\right].$$

Using (a) and the formula above, we then have

$$\begin{aligned}(q+1)\sigma'_{q+1,r+1}(A,B) &= (kr + l(q-r))T_{q,r}(A,B)^i_j A^j_i\phi + rT_{q,r}(A,B)^i_j M^j_i \\ &+ (q-r)T_{q,r+1}(A,B)^i_j N^j_i + T_{q,r}(A,B)^i_j(kA^j_i\phi + M^j_i).\end{aligned}$$

Using (a) again gives the result. □

Now we check that two definitions of $\mathcal{B}^k$'s, (3) and (4), coincide when the boundary is umbilic. By definition,

$$\sigma_{q,r}(A^T,\mu g) = \frac{(n-1-r)!}{q!(n-1-q)!}\sum_{i_1,\cdots,j_1\cdots<n}\left(\begin{array}{c} i_1 \cdots i_r \\ j_1 \cdots j_r \end{array}\right) A^{j_1}_{i_1}\cdots A^{j_r}_{i_r}\mu^{q-r}.$$

Therefore, $\sigma_{q,r}(A^T,\mu g) = \frac{r!(n-1-r)!}{q!(n-1-q)!}\sigma_r(A^T)\mu^{q-r}$ and $\sum_{i=0}^{k-1} C_1(n,k,i)\sigma_{2k-i-1,i}(A^T,\mu g) = \sum_{i=0}^{k-1}\frac{(n-1-i)!}{(n-k)!(2k-2i-1)!!}\sigma_i(A^T)\mu^{q-r} = \sum_{i=0}^{k-1} C_2(n,k,i)\sigma_i(A^T)\mu^{q-r}$.

Next, we show some properties of curvatures on the boundary. We review two of the fundamental equations: $R_{ijkl,m} + R_{ijmk,l} + R_{ijlm,k} = 0$ (Bianchi identity) and $R_{\alpha\beta\gamma n} = L_{\alpha\gamma,\beta} - L_{\beta\gamma,\alpha}$ (Codazzi equation), where n is the unit inner normal with respect to g. In Fermi (geodesic) coordinates, the metric is expressed as $g = dx^n dx^n + g_{\alpha\beta}dx^\alpha dx^\beta$. The Christoffel symbols satisfy

$$\Gamma^n_{\alpha\beta} = L_{\alpha\beta}, \quad \Gamma^\beta_{\alpha n} = -L_{\alpha\gamma}g^{\gamma\beta}, \quad \Gamma^n_{\alpha n} = 0 \tag{10}$$

on the boundary. When the boundary is umbilic, they become

$$\Gamma^n_{\alpha\beta} = \mu g_{\alpha\beta}, \quad \Gamma^\beta_{\alpha n} = -\mu\delta_{\alpha\beta}, \quad \Gamma^n_{\alpha n} = 0. \tag{11}$$

We denote the tensors and covariant differentiations with respect to the induced metric $g_{\alpha\beta}$ on the boundary by a *tilde* (e.g. $\tilde{R_{\alpha\beta}}$, $\mu_{\tilde{\alpha}\tilde{\beta}}$). Then the Christoffel symbols satisfy

$$\widetilde{\Gamma}^\gamma_{\alpha\beta} = \frac{1}{2}g^{\gamma\delta}\left(\frac{\partial g_{\alpha\delta}}{\partial x_\beta} + \frac{\partial g_{\beta\delta}}{\partial x_\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x_\delta}\right) = \Gamma^\gamma_{\alpha\beta}. \tag{12}$$

We also denote the Laplacian in the induced metric by $\widetilde{\Delta}$.

The next lemma gives us the relation between $\mathcal{B}^k_g$ and h_g.

Lemma 5. *Let (M,g) be a compact manifold with umbilic boundary. If $h_g = 0$ on the boundary, then we have $\mathcal{B}^k_g = 0$. Conversely, if $\mathcal{B}^k_g = 0$ on the boundary and if in addition $A_g \in \Gamma^+_k$, then $h_g = 0$.*

Proof. Let $L_{\alpha\beta} = \mu g_{\alpha\beta}$. By Definition 1, $\sigma_{2,1}(A^T, L) = \sigma_{2,1}(A^T, \mu g) = \frac{1}{2} tr T_1(A^T)\mu = \frac{n-2}{2}\sigma_1(A^T)\mu$. Therefore, when $n \geq 4$, we obtain $\mathcal{B}^2 = \sigma_1(A^T)\mu + \frac{2}{(n-2)(n-3)}\sigma_3(\mu g) = (\sigma_1(A^T) + \frac{n-1}{3}\mu^2)\mu$. When $n = 3$, $\mathcal{B}^2 = \sigma_1(A^T)\mu + \frac{1}{3}(2\mu)^3 - \frac{1}{2}(2\mu)(2\mu^2) = (\sigma_1(A^T) + \frac{2}{3}\mu^2)\mu$. For $k \geq 3$, we have $\mathcal{B}^k = (\sum_{i=0}^{k-1} C_2(n,k,i)\sigma_i(A^T)\mu^{2k-2i-2})\mu$, where $C_2(n,k,i)$ is positive.

Since $(n-1)\mu = h$, if $h = 0$, then clearly $\mathcal{B}_g^k = 0$. When $A_g \in \Gamma_k^+$, by Lemmas 2 and 3 we have $T_i(A)_n^n = \sigma_i(A^T)$ is positive for $i < k$. As a result, $\mathcal{B}_g^k = 0$ implies $h = 0$. □

We verify that the Schouten tensor satisfies conditions (T0)-(T2) when $\mathcal{W} = 0$ and $\mathcal{C} = 0$ on the boundary.

Lemma 6. *Suppose that the boundary is umbilic. Let n be the unit inner normal with respect to g. Then*

(a) $A_{\alpha n} = \mu_\alpha$ *on* ∂M;

(b) $\mu_{\tilde{\alpha}\tilde{\beta}} = A_{\alpha n,\beta} + A_{nn}\mu g_{\alpha\beta} - A_{\alpha\beta}\mu$ *on* ∂M;

(c) If $\mathcal{W} = 0$ *on* ∂M, *then we have* $R_{n\alpha n\beta} = A_{\alpha\beta} + A_{nn}g_{\alpha\beta}$ *on the boundary. If in addition* $\mathcal{C} = 0$ *on* ∂M, *then* $A_{\alpha\beta,n} - 2\mu A_{\alpha\beta} = \mu_{\tilde{\alpha}\tilde{\beta}} - R_{\alpha n\beta n}\mu$.

Proof. By the Codazzi equation, we get $R_{\alpha n} = (n-2)\mu_\alpha$ and $A_{\alpha n} = \mu_\alpha$.

For (b), we use (a), (11) and (12) to get

$$\mu_{\tilde{\alpha}\tilde{\beta}} = \partial_\beta A_{\alpha n} - \Gamma^\gamma_{\alpha\beta}\mu_\gamma = (A_{\alpha n,\beta} + \Gamma^l_{\alpha\beta}A_{ln} + \Gamma^l_{\beta n}A_{\alpha l}) - \Gamma^\gamma_{\alpha\beta}\mu_\gamma = A_{\alpha n,\beta} + A_{nn}\mu g_{\alpha\beta} - A_{\alpha\beta}\mu.$$

For (c), using the curvature decomposition formula $R_{ijkl} = \mathcal{W}_{ijkl} + A_{ik}g_{jl} + A_{jl}g_{ik} - A_{il}g_{jk} - A_{jk}g_{il}$, we first get $R_{n\alpha n\beta} = A_{nn}g_{\alpha\beta} + A_{\alpha\beta}$ when $\mathcal{W} = 0$. If in addition $\mathcal{C} = 0$, then $A_{\alpha\beta,n} - 2\mu A_{\alpha\beta} = A_{\alpha n,\beta} - 2\mu A_{\alpha\beta} = A_{\alpha n,\beta} + \mu A_{nn}g_{\alpha\beta} - A_{\alpha\beta}\mu - R_{\alpha n\beta n}\mu$. □

The next lemma will be used in proving Theorem 5 and 6.

Lemma 7. *Suppose ∂M is umbilic. Let u satisfy $u_n = -\mu + \hat{\mu}e^{-u}$, where $\hat{\mu}$ is constant. Then we have*

$$u_{n\alpha} = -\mu_\alpha + \mu u_\alpha - \hat{\mu}u_\alpha e^{-u}; \tag{13}$$

$$\begin{aligned} u_{\alpha\beta n} &= (2\mu - \hat{\mu}e^{-u})u_{\alpha\beta} - \mu u_{nn}g_{\alpha\beta} + \hat{\mu}u_\alpha u_\beta e^{-u} - \mu_{\tilde{\alpha}\tilde{\beta}} + \mu_\alpha u_\beta + \mu_\beta u_\alpha \\ &\quad -\mu_\gamma u_\gamma g_{\alpha\beta} + R_{n\beta\alpha n}(-\mu + \hat{\mu}e^{-u}) - \mu(-\mu + \hat{\mu}e^{-u})^2 g_{\alpha\beta}. \end{aligned} \tag{14}$$

Proof. By (11), $u_{n\alpha} = \partial_\alpha u_n - \Gamma^j_{\alpha n}u_j = \partial_\alpha u_n + \mu u_\alpha = -\mu_\alpha - \hat{\mu}u_\alpha e^{-u} + \mu u_\alpha$. For (14), by (11) and (12) $u_{n\alpha\beta} = \partial_\beta u_{n\alpha} - \Gamma^j_{\beta n}u_{j\alpha} - \Gamma^j_{\alpha\beta}u_{nj} = \partial_\beta u_{n\alpha} + \mu u_{\beta\alpha} - \tilde{\Gamma}^\gamma_{\alpha\beta}u_{n\gamma} - \mu u_{nn}g_{\alpha\beta}$. Now by (13), (11) and $u_n = -\mu + \hat{\mu}e^{-u}$,

$$\begin{aligned} u_{n\alpha\beta} &= u_{n\tilde{\alpha}\tilde{\beta}} + \mu u_{\beta\alpha} - \mu u_{nn}g_{\alpha\beta} \\ &= -\mu_{\tilde{\alpha}\tilde{\beta}} - \hat{\mu}u_{\alpha\beta}e^{-u} + \hat{\mu}u_\alpha u_\beta e^{-u} + \mu_\beta u_\alpha + 2\mu u_{\alpha\beta} - \mu u_{nn}g_{\alpha\beta} - \mu(-\mu + \hat{\mu}e^{-u})^2 g_{\alpha\beta}. \end{aligned}$$

On the other hand, using the Codazzi equation gives $u_{\alpha\beta n} = u_{n\alpha\beta} + R_{n\beta\alpha j}u_j = u_{n\alpha\beta} + \mu_\alpha u_\beta - \mu_\gamma u_\gamma g_{\alpha\beta} + R_{n\beta\alpha n}(-\mu + \hat{\mu}e^{-u})$. Combing above formulas yields (14). □

The last lemma of this section is a boundary version of the Bianchi identity.

Lemma 8. *Suppose that the boundary ∂M is umbilic and under a conformal change $\hat{g} = e^{-2u}g$, $\hat{L}_{\alpha\beta} = 0$ near a boundary point x_0. Then $g^{\alpha\beta}\hat{A}_{\alpha\beta,n} = 2\mu g^{\alpha\beta}\hat{A}_{\alpha\beta}$ at x_0.*

Proof. We denote the covariant differentiation with respect to the new metric $\hat{g}$ by $\hat{\nabla}$. Since $\hat{L}_{\alpha\beta} = 0$, by the Codazzi equation $\hat{R}_{\alpha\beta\gamma n} = 0$. Therefore, we have $\hat{R}_{\alpha n} = 0$ and $\hat{A}_{\alpha n} = 0$ at x_0. Hence, $\hat{\nabla}_\beta \hat{R}_{\alpha n} = \partial_\beta \hat{R}_{\alpha n} - \hat{\Gamma}^k_{\beta\alpha}\hat{R}_{kn} - \hat{\Gamma}^k_{\beta n}\hat{R}_{\alpha k} = -\hat{\Gamma}^n_{\beta\alpha}\hat{R}_{nn} - \hat{\Gamma}^\gamma_{\beta n}\hat{R}_{\alpha\gamma}$. By (11), both $\hat{\Gamma}^n_{\beta\alpha}$ and $\hat{\Gamma}^\gamma_{\beta n}$ are zero. Thus, we have $\hat{\nabla}_\beta \hat{R}_{\alpha n} = 0$.

On the other hand, by the Bianchi identity, $0 = \hat{\nabla}_n \hat{R}_{i\alpha k\beta} + \hat{\nabla}_k \hat{R}_{i\alpha\beta n} + \hat{\nabla}_\beta \hat{R}_{i\alpha n k}$. Contracting indices i and k gives $0 = \hat{\nabla}_n \hat{R}_{\alpha\beta} + \hat{g}^{ik}\hat{\nabla}_k \hat{R}_{i\alpha\beta n} - \hat{\nabla}_\beta \hat{R}_{\alpha n}$. Noting that $\hat{\nabla}_\beta \hat{R}_{\alpha n} = 0$ and $\hat{g}^{\alpha n} = 0$, contract indices α and β to get $0 = \hat{g}^{\alpha\beta}\hat{\nabla}_n \hat{R}_{\alpha\beta} - \hat{g}^{ik}\hat{\nabla}_k \hat{R}_{in} = \hat{g}^{\alpha\beta}\hat{\nabla}_n \hat{R}_{\alpha\beta} - \hat{g}^{nn}\hat{\nabla}_n \hat{R}_{nn}$. Therefore,

$$\hat{g}^{\alpha\beta}\hat{\nabla}_n \hat{A}_{\alpha\beta} = \frac{1}{(n-2)}(\hat{g}^{\alpha\beta}\hat{\nabla}_n \hat{R}_{\alpha\beta} - \frac{1}{2}\hat{R}_n) = \frac{1}{2(n-2)}(\hat{g}^{\alpha\beta}\hat{\nabla}_n \hat{R}_{\alpha\beta} - \hat{g}^{nn}\hat{\nabla}_n \hat{R}_{nn}) = 0. \quad (15)$$

Using $\hat{A}_{\alpha n} = 0$, $\hat{\Gamma}^n_{\alpha\beta} = \hat{\Gamma}^\alpha_{\beta n} = 0$ and (11), we finally arrive at

$$0 = \hat{g}^{\alpha\beta}\hat{\nabla}_n \hat{A}_{\alpha\beta} = \hat{g}^{\alpha\beta}\partial_n \hat{A}_{\alpha\beta} = \hat{g}^{\alpha\beta}(\hat{A}_{\alpha\beta,n} + \Gamma^k_{n\alpha}\hat{A}_{k\beta} + \Gamma^k_{n\beta}\hat{A}_{k\alpha}) = \hat{g}^{\alpha\beta}(\hat{A}_{\alpha\beta,n} - 2\mu\hat{A}_{\alpha\beta}).$$

□

2 Four-manifolds

In this section, we only consider $n = 4$. We prove Theorem 2 and Corollary 3. The proof of Theorem 2 consists of two propositions:

Proposition 1. *Let (M, g) be a compact connected four-manifold with umbilic boundary. If $Y(M, \partial M, [g])$ and $\int_M \sigma_2 + \frac{1}{2}\oint_{\partial M} \mathcal{B}_g$ are both positive, then there exists a metric $\hat{g} \in [g]$ such that $R_{\hat{g}} > 0$, $\sigma_2(A_{\hat{g}}) > 0$, and the boundary is totally geodesic.*

Proposition 2. *Suppose (M, g) is a compact connected four-manifold with totally geodesic boundary. If $R_g > 0$, $\sigma_2(A_g) > 0$, and (M, g) is not conformally equivalent to $(\mathbb{S}^+_4, g_c)$, then given a positive function f there exists a metric $\hat{g} \in [g]$ such that $\sigma_2(A_{\hat{g}}) = f$ and $\mathcal{B}_{\hat{g}}$ is zero.*

We will prove Propositions 1 and 2 in Subsections 2.1 and 2.2, respectively.

2.1 Conformal Metric Satisfying $\sigma_2 > 0$

We will deform a Yamabe metric to the one satisfying the properties in Proposition 1. The deformation comes from a nice idea by Gursky-Viaclovsky [18] for closed four-manifolds.

Proof of Proposition 1. Let the background metric g be a Yamabe metric. Thus, we have R_g is a positive constant and the boundary is totally geodesic.

Let $A^t = A + \frac{1-t}{2}(tr_g A)g$. Let $\hat{g} = e^{-2u}g$. For $n = 4$, the tensor $\hat{A}^t$ satisfies $\hat{A}^t = \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + \frac{1-t}{2}(\Delta u - |\nabla u|^2)g + A^t$. We can choose a large number Θ such that $A^{-\Theta} = \frac{1}{2}(Ric_g + \frac{\Theta}{6}R_g g)$ is positive definite. Let $f(x) = \sigma_2^{\frac{1}{2}}(A_g^{-\Theta})$. Thus, $A_g^{-\Theta} \in \Gamma_2^+$ and f is positive. Consider the following path of equations for $-\Theta \leq t \leq 1$:

$$\begin{cases} \sigma_2^{\frac{1}{2}}(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + \frac{1-t}{2}(\Delta u - |\nabla u|^2)g + A_g^t) = f(x)e^{2u} & in\, M \\ \frac{\partial u}{\partial n} = 0 & on\, \partial M. \end{cases} \tag{16}$$

Let $\mathcal{S} = \{t \in [-\Theta, 1] : \exists$ a solution $u \in C^{2,\alpha}(M)$ to (16) with $\hat{A}^t \in \Gamma_2^+\}$. At $t = -\Theta$, we have $u \equiv 0$ is a solution and $A_g^{-\Theta} \in \Gamma_2^+$. Therefore, $\mathcal{S}$ is nonempty. Consider the linearized operator $\mathcal{P}^t : C^{2,\alpha}(M) \cap \{\frac{\partial u}{\partial n} = 0 \text{ on } \partial M\} \to C^\alpha(M)$. It was proved in [18] (Proposition 2.2) that the linearized operator is elliptic with the strictly negative coefficient in the zeroth order term. By elliptic theory for Neumann condition [13], the linearized operator is invertible. Hence, $\mathcal{S}$ is open. If $\mathcal{S}$ is also closed, then we have a solution u to (16) at $t = 1$ with $\hat{A}^1 = \hat{A} \in \Gamma_2^+$. This gives $\hat{g} = e^{-2u}g$ satisfying $\sigma_2(\hat{A}) > 0$, $\hat{R} > 0$ and $\hat{\mu} = 0$. Thus, it remains to establish a priori estimates for solutions to (16) independent of t.

(1) C^0 estimates.

At the maximal point x_0 of u, if x_0 is in the interior, we have $|\nabla u| = 0$. If x_0 is at the boundary, since $\frac{\partial u}{\partial n} = 0$, we also have $|\nabla u| = 0$. Therefore, we get that $\nabla^2 u(x_0)$ is negative semi-definite and $\Delta u(x_0) \leq 0$. By Lemma 2 (c),

$$f(x_0)e^{2u(x_0)} = \sigma_2^{\frac{1}{2}}(g^{-1}\hat{A}^t) \leq \frac{\sqrt{6}}{4}\sigma_1(g^{-1}\hat{A}^t) = \frac{\sqrt{6}}{4}(3-2t)\Delta u + \frac{\sqrt{6}}{4}tr_g\, A_g^t \leq \frac{\sqrt{6}}{4}tr_g\, A_g^t \leq C,$$

where in the second inequality we use $t \leq 1$ and $\Delta u \leq 0$. Hence, u is upper bounded.

Now we prove the Harnack inequality. Let $H = |\nabla u|^2$. If the maximum of H is in the interior, then $\nabla H = 0$, and $\nabla^2 H$ is negative semi-definite. If the maximum of H is at the boundary, since $\frac{\partial u}{\partial n} = 0$ and $\mu_g = 0$, we have $u_{\alpha n} = 0$ and $H_n = 2u_\alpha u_{\alpha n} + 2u_n u_{nn} = 0$. Thus, we also have that $\nabla H = 0$, and $\nabla^2 H$ is negative semi-definite. Interior gradient estimates for (16) were proved in [18] (Proposition 4.1). We remark that the same proof works for boundary gradient estimates. The reason is that at the maximal point once we have $\nabla H = 0$, and $\nabla^2 H$ is negative semi-definite, then the rest of computations in [18] is the same regardless of the point being in the interior or on the boundary. Therefore, we get $|\nabla u| < C$. Thus, $\sup_M u \leq \inf_M u + C$.

To prove that $\sup_M u$ is lower bounded, integrating the equation gives

$$Ce^{4\sup_M u} \geq \int_M f^2 e^{4u} dV_g = \int_M \sigma_2(g^{-1}\hat{A}^t)dV_g = \int_M \sigma_2(\hat{g}^{-1}\hat{A}^t)dV_{\hat{g}},$$

where in the second equality we use $dV_{\hat{g}} = e^{-4u}dV_g$. Note that $\sigma_2(\hat{g}^{-1}\hat{A}^t) = \sigma_2(\hat{A}) + \frac{3}{2}(1-t)(2-t)\sigma_1^2(\hat{A})$. Thus, the above formula becomes

$$Ce^{4\sup_M u} \geq \int_M (\sigma_2(\hat{A}) + \frac{3}{2}(1-t)(2-t)\sigma_1^2(\hat{A}))dV_{\hat{g}} \geq \int_M \sigma_2(\hat{A})dV_{\hat{g}}.$$

Recall that the conformal invariant $\int_M \sigma_2 + \frac{1}{2}\oint_{\partial M} \mathcal{B}_g$ is positive. Since $\hat{\mu} = 0$, by Lemma 5 we get $\hat{\mathcal{B}} = 0$. Finally, we have

$$Ce^{4\sup_M u} \geq \int_M \sigma_2(\hat{A}) dV_{\hat{g}} + \frac{1}{2}\oint_{\partial M} \hat{\mathcal{B}}\, dS_{\hat{g}} = \int_M \sigma_2(A_g) dV_g + \frac{1}{2}\oint_{\partial M} \mathcal{B}_g\, dS_g > 0.$$

(2) C^2 estimates.

Interior C^2 estimates are proved in [6]. To get boundary C^2 estimates, we use Fermi coordinates in a tubular neighborhood $\partial M \times [0, \iota]$ of the boundary. Note that ∂M is compact so ι is a positive number. Thus, by Theorem 6 (b) (with $S = 0$) we obtain boundary C^2 estimates in each half ball $\overline{B}_r^+$. Since ∂M is compact, there are finitely many local charts of the tubular neighborhood. We then get the required estimates.

(3) C^∞ estimates.

Once we have C^2 bounds, the equation is uniformly elliptic and concave. Higher order regularity follows by standard elliptic theories; see [10],[20] and [23]. $\square$

2.2 Conformal Metric Satisfying $\sigma_2 = f$

In this subsection, we proof Proposition 2. We first prove a lemma.

Lemma 9. *Let (M, g) be a compact four-manifold with umbilic boundary. Suppose $Y(M, \partial M, [g]) > 0$. Then*

$$\int_M \sigma_2(A_g) + \frac{1}{2}\oint_{\partial M} \mathcal{B}_g \leq 2\pi^2.$$

Moreover, the equality holds if and only if (M, g) is conformally equivalent to $(\mathbb{S}^4_+, g_c)$, where g_c is the standard metric on the hemisphere.

Proof. Denote the volume of (M, g) by V_g. Let $\tilde{g}$ be a Yamabe metric such that $R_{\tilde{g}}$ is constant and the boundary is totally geodesic. It was proved by Escobar [9] that

$$Y(M, \partial M, g) = \frac{\int_M R_{\tilde{g}} + \oint_{\partial M} 3\mu_{\tilde{g}}}{V_{\tilde{g}}^{\frac{1}{2}}} = R_{\tilde{g}} V_{\tilde{g}}^{\frac{1}{2}} \leq Y(\mathbb{S}^4_+, \mathbb{S}^3, g_c) = 8\sqrt{3}\pi. \tag{17}$$

The equality holds if and only if (M, g) is conformally equivalent to $(\mathbb{S}^4_+, g_c)$. Since $\mu_{\tilde{g}} = 0$, by Lemma 5 we have $\mathcal{B}_{\tilde{g}} = 0$. Therefore, $\int_M \sigma_2(A_g) + \frac{1}{2}\oint_{\partial M} \mathcal{B}_g = \int_M \sigma_2(A_{\tilde{g}}) + \frac{1}{2}\oint_{\partial M} \mathcal{B}_{\tilde{g}} = \int_M \sigma_2(A_{\tilde{g}})$. Note that $\sigma_2(A) = \frac{1}{8}(\frac{1}{12}R^2 - |E|^2)$, where $E = Ric - \frac{1}{4}Rg$. By (17) we get $\int_M \sigma_2(A_g) + \frac{1}{2}\oint_{\partial M} \mathcal{B}_g = \frac{1}{8}\int_M (\frac{1}{12}R_{\tilde{g}}^2 - |E_{\tilde{g}}|^2) \leq \frac{1}{96}R_{\tilde{g}}^2 V_{\tilde{g}} \leq 2\pi^2$. The equality holds if and only if (M, g) is conformally equivalent to $(\mathbb{S}^4_+, g_c)$. $\square$

Proof of Proposition 2. Let $\hat{A} = \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g$. Since $\mu_g = 0$, the problem is equivalent to solve

$$\begin{cases} \sigma_2^{\frac{1}{2}}(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g) = f(x)\, e^{-2u} & \textit{in } M \\ \frac{\partial u}{\partial n} = 0 & \textit{on } \partial M \end{cases}$$

with $\hat{A} \in \Gamma_2^+$.

Denote the volume of (M, g) by V_g. We will use a deformation motivated by [19], [17] for closed manifolds. Let $S_g = (1 - \zeta(t))(\frac{1}{\sqrt{6}} V^{\frac{2}{5}} g - A_g)$. Consider the following path of equations for $0 \le t \le 1$ with $\hat{A} + S_g \in \Gamma_2^+$:

$$\begin{cases} \sigma_2^{\frac{1}{2}}(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g + S_g) = (1-t)(\int_M e^{-5u})^{\frac{2}{5}} + \zeta(t) f\, e^{-2u} & \text{in } M \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial M, \end{cases} \tag{18}$$

where $\zeta(t) \in C^1[0,1]$ satisfies $0 \le \zeta \le 1$, $\zeta(0) = 0$, and $\zeta = 1$ for $t \ge \frac{1}{2}$. The Leray-Schauder degree is defined by considering the space $\{u \in C^{4,\alpha}(M) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial M\}$; see [7]. We check that at $t = 0$ the degree is nonzero. For closed manifolds, it was proved in [19] that the degree is nonzero at $t = 0$. For manifolds with boundary, we remark that the same proof works. More specifically, at $t = 0$, (18) becomes

$$\begin{cases} \sigma_2^{\frac{1}{2}}(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + \frac{1}{\sqrt{6}} V^{\frac{2}{5}} g) = (\int_M e^{-5u})^{\frac{2}{5}} & \text{in } M \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial M. \end{cases}$$

By the boundary condition $\frac{\partial u}{\partial n} = 0$ if the maximum (resp. minimum) of u happens at the boundary, we still have $\nabla u = 0$, and $\nabla^2 u$ is negative (resp. positive) semi-definite. Hence, as in [19] by the maximum principle, $u = 0$ is the unique solution.

Now the linearized operator $\mathcal{P} : C^{2,\alpha}(M) \cap \{\frac{\partial \phi}{\partial n} = 0 \text{ on } \partial M\} \to C^\alpha(M)$ at $u = 0$ is $\mathcal{P}(\phi) = \frac{\sqrt{6}}{4} \Delta \phi + 2 V_g^{-\frac{2}{5}} \int_M \phi$. Then the rest of the proof of showing the degree is nonzero at $t = 0$ follows from [19]. Consequently, the problem reduces to establishing a priori estimates for (18).

Suppose we have uniform C^0 bounds for (18). By [6], we get interior C^2 estimates. For boundary C^2 estimates, we check that S satisfies the condition in Theorem 6 (b). Since $\mu = 0$, by Lemma 6 (a) we have $S_{\alpha n} = 0$ and by (15) $g^{\alpha\beta} S_{\alpha\beta,n} = -(1-\zeta(t)) g^{\alpha\beta} A_{\alpha\beta,n} = 0$. Hence, we have boundary C^2 estimates in each half ball $\overline{B}_r^+$ in Fermi coordinates. Thus, higher order regularities. It remains to derive a priori C^0 estimates. We begin by proving the boundedness of the integral term in (18).

Lemma 10. *Let u be a solution to (18) with $t \in [0, 1]$. Then $(1-t)(\int_M e^{-5u})^{\frac{2}{5}} < C$.*

Proof. Since $\frac{\partial u}{\partial n} = 0$ on the boundary, at the maximum point x_0, we have $\nabla u = 0$ and $\nabla^2 u$ is negative semi-definite, no matter x_0 being in the interior or at the boundary. Thus,

$$(1-t)(\int_M e^{-5u})^{\frac{2}{5}} \le \sigma_2^{\frac{1}{2}}(\nabla^2 u(x_0) + A_g(x_0) + S_g(x_0)) \le \sigma_2^{\frac{1}{2}}(A_g(x_0) + S_g(x_0)) < C.$$

□

Now we prove that $\inf_M u > -C$.

(1) $\inf_M u > -C$ for $t \in [0, 1 - \varepsilon]$.

The analysis depends on whether the infimum point is close to the boundary or not. Suppose there is a sequence of solutions $\{u^i\}$ to (18) with $t = t_i \leq 1 - \varepsilon$ such that $u^i(p_i) = \inf u^i \to -\infty$ and $p_i \to p_0$. Let $\epsilon_i = e^{\inf u^i} \to +0$ and d_i be the distance from p_i to ∂M. We will show that there is a contradiction.

Case a. Non-tangential approach. Assume $\frac{d_i}{\epsilon_i} \to +\infty$.
Using the normal coordinates at p_i, we define the mapping

$$\begin{aligned} \mathcal{T}_i : B(0, \tfrac{d_i}{\epsilon_i}) \subset \mathbb{R}^4 \to \quad & M \\ x \to \quad & \exp_{p_i}(\epsilon_i x) = y. \end{aligned}$$

On $\mathbb{R}^4$, define the metric $g_i = \epsilon_i^{-2}\mathcal{T}_i^* g$ and the function $\tilde{u}^i = u^i(\mathcal{T}_i(x)) - \ln \epsilon_i$. Then $\tilde{u}^i(0) = u^i(p_i) - \ln \epsilon_i = 0$ and $\tilde{u}^i(x) \geq 0$. Moreover,

$$\sigma_2^{\frac{1}{2}}(\nabla_{g_i}^2 \tilde{u}^i + d\tilde{u}^i \otimes_{g_i} d\tilde{u}^i - \frac{1}{2}|\nabla_{g_i}\tilde{u}^i|^2 g_i + A_{g_i} + S_{g_i}) = \epsilon_i^2(1-t_i)(\int_M e^{-5u^i})^{\frac{2}{5}} + \zeta(t_i) f(\mathcal{T}_i(x))\, e^{-2\tilde{u}^i}$$

on $B(0, \frac{d_i}{\epsilon_i})$ in $\mathbb{R}^4$. Note that g_i tends to the Euclidean metric ds^2 as i goes to infinity. By Lemma 10, the integral term is bounded. Hence, by local estimates [6] and the fact that $\tilde{u}^i \geq 0$, we get $\sup_{B(0,r)} |\nabla_{g_i}\tilde{u}^i| < C(r)$. Integrating from zero, we have $\sup_{B(0,1)} \tilde{u}^i < C$. On the other hand, since $t \leq 1 - \varepsilon$ for a fixed number ε, by Lemma 10

$$\int_{B(0,1)} e^{-5\tilde{u}^i} dV_{g_i} = \epsilon_i \int_{B(p_i,\epsilon_i)} e^{-5u^i} dV_g < \epsilon_i C \to 0$$

as $i \to \infty$. This contradicts to $\sup_{B(0,1)} \tilde{u}^i < C$.

Case b. Tangential approach. Assume $\frac{d_i}{\epsilon_i} \leq C_0$ for some fixed number C_0.
Let p_i' be a point on the boundary such that the distance between p_i and p_i' is d_i. We may assume the Fermi coordinates are defined in a tubular neighbor of length κ. Around the point p_i', we define the mapping

$$\begin{aligned} U_i : \overline{B}_+(0, \tfrac{\kappa}{\epsilon_i}) \subset \overline{\mathbb{R}}^4_+ \to \quad & M \\ x \to \quad & G_{p_i'}(\epsilon_i x) = y, \end{aligned}$$

where G is the normal exponential map; see [8]. We may assume that $\frac{d_i}{\epsilon_i} < \frac{\kappa}{\epsilon_i}$. On $\overline{\mathbb{R}}^4_+$, define the metric $g_i = \epsilon_i^{-2} U_i^* g$ and the function $\tilde{u}^i = u^i(U_i(x)) - \ln \epsilon_i$. Let $q_i \in \overline{B}_+(0, \frac{\kappa}{\epsilon_i})$ be the point satisfying $U_i(q_i) = p_i$. Therefore, $q_i \in \overline{B}_+(0, \frac{d_i}{\epsilon_i}) \subset \overline{B}_+(0, C_0)$ belongs to a compact subset in $\overline{\mathbb{R}}^4_+$. We have $\tilde{u}^i(q_i) = u^i(p_i) - \ln \epsilon_i = 0$ and $\tilde{u}^i(x) \geq 0$. Moreover,

$$\sigma_2^{\frac{1}{2}}(\nabla_{g_i}^2 \tilde{u}^i + d\tilde{u}^i \otimes_{g_i} d\tilde{u}^i - \frac{1}{2}|\nabla_{g_i}\tilde{u}^i|^2 g_i + A_{g_i} + S_{g_i}) = \epsilon_i^2(1-t_i)(\int_M e^{-5u^i})^{\frac{2}{5}} + \zeta(t_i) f(U_i(x))\, e^{-2\tilde{u}^i}$$

on $\overline{B}_+(0, \frac{\kappa}{\epsilon_i})$ in $\overline{\mathbb{R}}^4_+$ with $\frac{\partial \tilde{u}^i}{\partial n} = 0$ on $\overline{B}_+(0, \frac{\kappa}{\epsilon_i}) \cap \{x_4 = 0\}$.

Using Theorem 6 (a), Lemma 10 and the fact that $\tilde{u}^i \geq 0$, we get $\sup_{\overline{B}_+(0,r)} |\nabla_{g_i} \tilde{u}^i| < C(r)$. Integrating from q_i, we have $\sup_{\overline{B}_+(0,C_0)} \tilde{u}^i < C$. On the other hand, since $t \leq 1 - \varepsilon$, by Lemma 10

$$\int_{\overline{B}_+(0,C_0)} e^{-5\tilde{u}^i} dV_{g_i} = \epsilon_i \int_{U_i(\overline{B}_+(0,C_0))} e^{-5u^i} dV_g < \epsilon_i C \to 0$$

as $i \to \infty$. This contradicts to $\sup_{\overline{B}_+(0,C_0)} \tilde{u}^i < C$.

(2) $\inf_M u > -C$ when $t \to 1$.

Suppose on the contrary there is a sequence of solutions $\{u^i\}$ with $t_i \to 1$ such that $u^i(p_i) = \inf u^i \to -\infty$ and $p_i \to p_0$. Let $\epsilon_i = e^{\inf u^i} \to +0$ and d_i be the distance from p_i to ∂M. For simplicity, we denote $e^{-2u^i} g$ by $\hat{g}_i$ and $A_{\hat{g}_i}$ by $\hat{A}_i$.

Case a. Non-tangential approach. Assume $\frac{d_i}{\epsilon_i} \to +\infty$.
Let $\mathcal{T}_i$, g_i, and $\tilde{u}^i$ be as in (1) Case a. Denote the metric $e^{-2\tilde{u}^i} g_i$ by $\tilde{g}_i$. Then $\tilde{u}^i(0) = u^i(p_i) - \ln \epsilon_i = 0$ and $\tilde{u}^i(x) \geq 0$. Moreover, since $t_i \to 1$, we have $\zeta(t_i) = 1$. Therefore,

$$\sigma_2^{\frac{1}{2}}(\nabla_{g_i}^2 \tilde{u}^i + d\tilde{u}^i \otimes_{g_i} d\tilde{u}^i - \frac{1}{2}|\nabla_{g_i} \tilde{u}^i|^2 g_i + A_{g_i}) = \epsilon_i^2 (1 - t_i) (\int_M e^{-5u^i})^{\frac{2}{5}} + f(\mathcal{T}_i(x))\, e^{-2\tilde{u}^i}$$

on $B(0, \frac{d_i}{\epsilon_i})$ in $\mathbb{R}^4$. Similar to (1) Case a, we get $\sup_{B(0,r)} |\tilde{u}^i| + |\nabla_{g_i} \tilde{u}^i|^2 + |\nabla_{g_i}^2 \tilde{u}^i| < C(r)$.

Now since $f(\mathcal{T}_i(x)) \to f(p_0)$, the equation is uniform elliptic and concave. Notice that $B(0, \frac{d_i}{\epsilon_i}) \to \mathbb{R}^4$. Therefore, $\{\tilde{u}^i\}$ converges uniformly on compact sets to a solution $u \in C^\infty(\mathbb{R}^4)$ of $\sigma_2^{\frac{1}{2}}(\nabla^2 u + du \otimes u - \frac{1}{2}|\nabla u|^2 ds^2) = f(p_0)\, e^{-2u}$. By the uniqueness theorem [4], $e^{-2u} ds^2$ comes from the pulling-back of the standard metric g_c on the sphere. Hence,

$$4\pi^2 \leftarrow \int_{B(0,\frac{d_i}{\epsilon_i})} \sigma_2(A_{\tilde{g}_i}) dV_{\tilde{g}_i} = \int_{B(p_i,d_i)} \sigma_2(\hat{A}_i) dV_{\hat{g}_i} \leq \int_M \sigma_2(\hat{A}_i) dV_{\hat{g}_i}.$$

On the other hand, since $\mu_{\hat{g}_i} = 0$, by Lemma 5 we have $\mathcal{B}_{\hat{g}_i} = 0$. Thus, by Lemma 9 $\int_M \sigma_2(\hat{A}_i) dV_{\hat{g}_i} = \int_M \sigma_2(\hat{A}_i) dV_{\hat{g}_i} + \frac{1}{2} \oint_{\partial M} \mathcal{B}_{\hat{g}_i} \, d\Sigma_{\hat{g}_i} \leq 2\pi^2$. This gives a contradiction.

Case b. Tangential approach. Assume $\frac{d_i}{\epsilon_i} \leq C_0$ for some fixed number C_0.
Let p_i' and κ be as in (1) Case b. We may assume that $\frac{d_i}{\epsilon_i} < \frac{\kappa}{\epsilon_i}$. Let U_i, g_i, q_i and $\tilde{u}^i$ be also as in (1) Case b. Denote the metric $e^{-2\tilde{u}^i} g_i$ by $\tilde{g}_i$. $q_i \in \overline{B}_+(0, \frac{d_i}{\epsilon_i}) \subset \overline{B}_+(0, C_0)$ belongs to a compact subset in $\overline{\mathbb{R}}^4_+$. We have $\tilde{u}^i(q_i) = u^i(p_i) - \ln \epsilon_i = 0$ and $\tilde{u}^i(x) \geq 0$. Moreover,

$$\sigma_2^{\frac{1}{2}}(\nabla_{g_i}^2 \tilde{u}^i + d\tilde{u}^i \otimes_{g_i} d\tilde{u}^i - \frac{1}{2}|\nabla_{g_i} \tilde{u}^i|^2 g_i + A_{g_i}) = \epsilon_i^2 (1 - t_i) (\int_M e^{-5u^i})^{\frac{2}{5}} + f(U_i(x))\, e^{-2\tilde{u}^i}$$

on $\overline{B}_+(0, \frac{\kappa}{\epsilon_i})$ in $\overline{\mathbb{R}}^4_+$ with $\frac{\partial \tilde{u}^i}{\partial n} = 0$ on $\overline{B}_+(0, \frac{\kappa}{\epsilon_i}) \cap \{x_4 = 0\}$. Similar to (1) Case b, by Theorem 6 (a), we get $\sup_{\overline{B}_+(0,r)} |\tilde{u}^i| + |\nabla_{g_i} \tilde{u}^i| < C(r)$. Then by Theorem 6 (b), we arrive at $\sup_{\overline{B}_+(0,r)} |\tilde{u}^i| + |\nabla_{g_i} \tilde{u}^i| + |\nabla_{g_i}^2 \tilde{u}^i| < C(r)$.

Now $\{\tilde{u}^i\}$ converges uniformly on compact sets to a solution $u \in C^\infty(\overline{\mathbb{R}}^4_+)$ of $\sigma_2^{\frac{1}{2}}(\nabla^2 u + du \otimes u - \frac{1}{2}|\nabla u|^2 ds^2) = f(p_0)\, e^{-2u}$ with $\frac{\partial u}{\partial n} = 0$ on $\{x_4 = 0\}$. By reflection, u extends to a

$C^{2,\alpha}$ solution of the above equation in $\mathbb{R}^4$. Further regularities give $u \in C^\infty(\mathbb{R}^4)$. By the uniqueness theorem, $e^{-2u}ds^2$ comes from the pulling-back of g_c. Hence,

$$2\pi^2 \leftarrow \int_{\overline{B}_+(0,\frac{\kappa}{\epsilon_i})} \sigma_2(A_{\tilde{g}_i}) dV_{\tilde{g}_i} = \int_{U_i(\overline{B}_+(0,\frac{\kappa}{\epsilon_i}))} \sigma_2(\hat{A}_i) dV_{\hat{g}_i} \leq \int_M \sigma_2(\hat{A}_i) dV_{\hat{g}_i}.$$

On the other hand, since $\mu_{\hat{g}_i} = 0$, by Lemma 5 we have $\mathcal{B}_{\hat{g}_i} = 0$. Thus, by Lemma 9 and the assumption that (M, g) is not conformally equivalent to the hemisphere, we finally arrive at $\int_M \sigma_2(\hat{A}_i) dV_{\hat{g}_i} = \int_M \sigma_2(\hat{A}_i) dV_{\hat{g}_i} + \frac{1}{2}\oint_{\partial M} \mathcal{B}_{\hat{g}_i}\, d\Sigma_{\hat{g}_i} < 2\pi^2$. This gives a contradiction.

(3) C^0 estimates.

Once u has a lower bound, by [6] and Theorem 6 (a) we have $|\nabla u| < C$. Thus, we obtain $\sup_M u \leq \inf_M u + C$. It remains to prove that $\inf_M u$ is upper bounded. Since $\frac{\partial u}{\partial n} = 0$ on the boundary, at the minimun point x_0, we have $\nabla u = 0$ and $\nabla^2 u$ is positive semi-definite, no matter x_0 being in the interior or at the boundary. Therefore,

$$Ce^{-2\inf u} \geq (1-t)(\int_M e^{-5u})^{\frac{2}{5}} + \zeta(t) f(x_0) e^{-2u} = \sigma_2^{\frac{1}{2}}(\nabla^2 u(x_0) + A_g(x_0)) \geq \sigma_2^{\frac{1}{2}}(A_g(x_0)) > 0.$$

□

2.3 Application to Einstein manifolds

In this subsection, we give an application of Theorem 1 to conformally compact Einstein manifolds.

Definition 2. *Let X^4 be a compact manifold with boundary $\partial X = N^3$ and g be a complete Einstein metric defined in the interior of X. (X, g) is called a conformally compact Einstein manifold if there exists a smooth defining function s for N such that $(X, s^2 g)$ is a compact Riemannian manifold with boundary.*

Each defining function induces a metric $s^2 g|_N = g_0$ on N. Thus (X, g) determines a conformal structure $(N^3, [g_0])$ called the *conformal infinity.* The *renormalized volume* $\mathcal{V}$ is a invariant of (X, g) coming from the volume expansion $Vol(\{s > \epsilon\}) = c_0\epsilon^{-3} + c_2\epsilon^{-1} + \mathcal{V} + o(1)$.

Corollary 3. *Let (X^4, g) be a conformally compact Einstein manifold with conformal infinity $(N^3, [g_0])$. Suppose that $Y(N^3, [g_0])$ and the renormalized volume $\mathcal{V}$ are both positive. Then there exists a conformal compactification $(X, \rho^2 g)$ such that $\sigma_2(A_{\rho^2 g})$ is a positive constant and the boundary is totally geodesic. Moreover, ρ is a defining function for N.*

Proof. First, Qing [24], [5] proved that if $Y(N^3, [g_0]) > 0$, then there exists a conformal compactification $(X, e^{-2u}g)$ such that $R_{e^{-2u}g}$ is positive and the boundary is totally geodesic. Denote this metric by $g_1 = e^{-2u}g$. Hence, we have

$$Y(X, N, [g_1]) > 0. \tag{19}$$

Secondly, for conformally compact Einstein four-manifolds, Andersen [1] proved that $32\pi^2\chi(X) = \int_X |\mathcal{W}|^2 dV_g + 4\mathcal{V}$. Now recall the Gauss-Bonnet formula for compact four-manifolds with boundary: $32\pi^2\chi(X,\partial X) = \int_X |\mathcal{W}|^2 + 16(\int_X \sigma_2(A_{g_1}) + \frac{1}{2}\oint_{\partial X}\mathcal{B}_{g_1})$. Since the boundary is totally geodesic $h_{g_1} = 0$, by Lemma 5 we have $\mathcal{B}_{g_1} = 0$. This gives

$$4\int_X \sigma_2(A_{g_1}) = \mathcal{V} > 0. \tag{20}$$

(19) and (20) then verify the conditions of Theorem 1. Therefore, by Theorem 1 there is a conformal metric $g_2 = e^{-2v}g_1$ such that $\sigma_2(A_{g_2})$ is a positive constant and the boundary is totally geodesic. Thus, $(X, g_2 = \rho^2 g)$ with $\rho = e^{-(u+v)}$ is a conformal compactification satisfying the properties required in the corollary. Moreover, since e^{-u} is a defining function (see [24]), it follows that $e^{-(u+v)}$ is also a defining function. $\square$

3 Functionals $\mathcal{F}_k$

In this section, we prove Theorem 3 and Corollary 1. We first prove a lemma.

Lemma 11. *Let A_g and L be the Schouten tensor and the second fundamental form, respectively. When the Cotten tensor is zero (i.e., $A_{ij,k} = A_{ik,j}$) or when $q = 1$, we have*

(a) $T_q(A)^i_{j,i} = 0$. (i.e., T_q is divergence free.)

Moreover, if (M,g) is locally conformally flat, we also have

(b) $T_{q,r}(A^T,L)^\alpha_{\beta,\tilde{\alpha}} = \frac{(n-1-q)(q-r)}{q} T_{q-1,r}(A^T,L)^\alpha_\beta A^n_\alpha - rT_{q,r-1}(A^T,L)^\alpha_\beta A^n_\alpha$;

(c) $T_q(A^T)^\alpha_{\beta,\tilde{\alpha}} = -qT_{q,q-1}(A^T,L)^\alpha_\beta A^n_\alpha$.

Proof. When (M,g) is locally conformally flat, (a) was proved in [28]; see also [3] for $q = 1$ case. Suppose (a) is true for $q < m$. By the recursive formula and $A_{ij,k} = A_{ik,j}$,

$$T_m(A)^i_{j,i} = \sigma_m(A)_{,i}g^i_j - T_{m-1}(A)^i_{k,i}A^k_j - T_{m-1}(A)^i_k A^k_{j,i} = \sigma_m(A)_{,j} - T_{m-1}(A)^i_k A^k_{j,i} = 0.$$

For (b), we first compute

$$\begin{aligned} T_{q,r}(A^T,L)^\alpha_{\beta,\alpha} &= \frac{1}{q!}\sum_{i_1,\cdots,j_1\cdots<n} \begin{pmatrix} i_1\cdots i_r\cdots i_q\,\alpha \\ j_1\cdots j_r\cdots j_q\,\beta \end{pmatrix} \times \\ &\quad \left[rA^{j_1}_{i_1}\cdots A^{j_r}_{i_r,\alpha}L^{j_{r+1}}_{i_{r+1}}\cdots L^{j_q}_{i_q} + (q-r)A^{j_1}_{i_1}\cdots A^{j_r}_{i_r}L^{j_{r+1}}_{i_{r+1}}\cdots L^{j_q}_{i_q,\alpha}\right] \\ &= \frac{1}{q!}\sum_{i_1,\cdots,j_1\cdots<n}(q-r)\begin{pmatrix} i_1\cdots i_r\cdots i_q\,\alpha \\ j_1\cdots j_r\cdots j_q\,\beta \end{pmatrix} A^{j_1}_{i_1}\cdots A^{j_r}_{i_r}L^{j_{r+1}}_{i_{r+1}}\cdots L^{j_q}_{i_q,\alpha}, \end{aligned}$$

where in the first equality, the first term is zero because $A^{j_r}_{i_r,\alpha}$ is symmetric in $(i_r\alpha)$. By the Codazzi equation and the curvature decomposition, we have $L_{\alpha\gamma,\beta} - L_{\beta\gamma,\alpha} = R_{\alpha\beta\gamma n} = A_{\beta n}g_{\alpha\gamma} - A_{\alpha n}g_{\beta\gamma}$. Therefore,

$$T_{q,r}(A^T,L)^\alpha_{\beta,\alpha} = \tfrac{q-r}{q!}\textstyle\sum_{i_1,\cdots,j_1\cdots<n}\begin{pmatrix} i_1\cdots i_q\,\alpha \\ j_1\cdots j_q\,\beta \end{pmatrix} A^{j_1}_{i_1}\cdots A^{j_r}_{i_r}L^{j_{r+1}}_{i_{r+1}}\cdots L^{j_{q-1}}_{i_{q-1}}g^{j_q}_{i_q}A_{\alpha n}.$$

Hence,

$$T_{q,r}(A^T,L)^\alpha_{\beta,\alpha} = \frac{q-r}{q!}(n-q-1)\,T_{q-1,r}(A^T,L)^\alpha_\beta A_{\alpha n}. \tag{21}$$

By definition, $\nabla_\gamma A^\beta_\alpha = \nabla_{\tilde{\gamma}} A^\beta_\alpha - L_{\alpha\gamma}A^\beta_n - L^\beta_\gamma A^n_\alpha$. Thus, we obtain

$$\begin{aligned}
&T_{q,r}(A^T,L)^\alpha_{\beta,\tilde{\alpha}} = T_{q,r}(A^T,L)^\alpha_{\beta,\alpha} \\
&+\frac{1}{q!}\sum_{i_1,\cdots,j_1\cdots<n}\begin{pmatrix} i_1\cdots i_r\cdots i_q\,\alpha \\ j_1\cdots j_r\cdots j_q\,\beta\end{pmatrix} rA^{j_1}_{i_1}\cdots A^{j_{r-1}}_{i_{r-1}}(L_{i_r\alpha}A^{j_r}_n + L^{j_r}_\alpha A^n_{i_r})L^{j_{r+1}}_{i_{r+1}}\cdots L^{j_q}_{i_q} \\
&= T_{q,r}(A^T,L)^\alpha_{\beta,\alpha} + \frac{r}{q!}\sum_{i_1,\cdots,j_1\cdots<n}\begin{pmatrix} i_1\cdots i_q\,\alpha \\ j_1\cdots j_q\,\beta\end{pmatrix} A^{j_1}_{i_1}\cdots A^{j_{r-1}}_{i_{r-1}}L^{j_r}_\alpha A^n_{i_r}L^{j_{r+1}}_{i_{r+1}}\cdots L^{j_q}_{i_q},
\end{aligned}$$

where in the first equality, the first term is zero because L is symmetric. Exchanging i_r and α, we arrive at

$$T_{q,r}(A^T,L)^\alpha_{\beta,\tilde{\alpha}} = T_{q,r}(A^T,L)^\alpha_{\beta,\alpha} - rT_{q,r-1}(A^T,L)^{i_r}_\beta A^n_{i_r}. \tag{22}$$

Combining (21) and (22) gives (b).

(c) follows from (b) by letting $r=q$. $\square$

In the following proof, for simplicity $\int$ stands for $\int_M$ and $\oint$ stands for $\oint_{\partial M}$.

Proof of Theorem 3. Let $g_t = e^{-2u_t}g$ be a conformal variation of g such that $u_0 = 0$. Suppose $u'_t = \phi$ at $t=0$. Then $g' = -2\phi g$ and $(g^{-1})' = 2\phi g^{-1}$. Consequently, $dV' = -n\phi dV$ and $d\Sigma' = -(n-1)\phi d\Sigma$.

By conformal change formulas of A_g and L, we get directly that $A'_{ij} = \phi_{ij}$ and $L'_{\alpha\beta} = -L_{\alpha\beta}\phi + \phi_n g_{\alpha\beta}$. Therefore, by raising indices we obtain

$$A'^j_i = A'_{im}g^{mj} + A_{im}g'^{mj} = \phi^j_i + 2\phi A^j_i \tag{23}$$

and

$$L'^\beta_\alpha = L'_{\alpha\gamma}g^{\gamma\beta} + L_{\alpha\gamma}g'^{\gamma\beta} = L^\beta_\alpha\phi + \phi_n g^\beta_\alpha. \tag{24}$$

Then by Lemma 4, we have

$$\begin{aligned}
\sigma'_{q+1,r+1}(A^T,L) &= (r+q+2)\sigma_{q+1,r+1}(A^T,L)\phi + \frac{r+1}{q+1}T_{q,r}(A^T,L)^\alpha_\beta\phi^\beta_\alpha \\
&\quad + \frac{(q-r)}{q+1}(n-1-q)\sigma_{q,r+1}(A^T,L)\phi_n,
\end{aligned} \tag{25}$$

$$\sigma'_{q+1}(A) = 2(q+1)\sigma_{q+1}(A)\phi + T_q(A)^i_j\phi^j_i, \tag{26}$$

$$\sigma'_{q+1}(L) = (q+1)\sigma_{q+1}(L)\phi + (n-1-q)\sigma_q(L)\phi_n. \tag{27}$$

By Lemma 11 (a), $T_q(A)$ is divergence free. Applying the integration by parts gives

$$\left(\int\sigma_k(A)\right)' = \int\sigma'_k(A)dV + \sigma_k(A)dV' = (2k-n)\int\sigma_k(A)\phi - \oint T_{k-1}(A)^n_j\phi^j, \tag{28}$$

where n is the unit inner normal.

(a) By (25) and Lemma 3 (c),

$$\left(\oint \sigma_{2,1}(A^T,L)\right)' = \oint\{(4-n)\phi\sigma_{2,1}(A^T,L)+\frac{1}{2}T_1(L)^\alpha_\beta\phi^\beta_\alpha+\frac{n-2}{2}\sigma_1(A^T)\phi_n\}$$
$$=\oint\{(4-n)\phi\sigma_{2,1}(A^T,L)+\frac{1}{2}T_1(L)^\alpha_\beta\phi^\beta_\alpha+\frac{n-2}{2}T_1(A)^n_n\phi_n\}. \tag{29}$$

For the second term in the last integral, applying integration by parts we get

$$\oint T_1(L)^\alpha_\beta\phi^\beta_\alpha=\oint T_1(L)^\alpha_\beta(\phi^{\tilde\beta}_{\tilde\alpha}-L^\beta_\alpha\phi_n)=\oint\{-T_1(L)^\alpha_{\beta,\alpha}\phi^\beta-2\sigma_2(L)\phi_n\}, \tag{30}$$

where in the last equality we use Lemma 4(a), and the fact that $L_{\alpha\beta,\gamma}=L_{\alpha\beta,\tilde\gamma}$ since the boundary is of codimension one. On the other hand, by the Codazzi equation, we have $R_{\beta n}=-L^\gamma_{\beta,\gamma}+h_{,\beta}$. Therefore, we get $T_1(A)^n_\beta=-A^n_\beta=-\frac{1}{n-2}R^n_\beta=\frac{1}{n-2}(L^\gamma_{\beta,\gamma}-h_{,\beta})$. As a result, we have the relation $T_1(L)^\alpha_{\beta,\alpha}=h_{,\alpha}g^\alpha_\beta-L^\alpha_{\beta,\alpha}=h_{,\beta}-L^\alpha_{\beta,\alpha}=-(n-2)T_1(A)^n_\beta$. Combining this relation, (29) and (30) gives

$$\left(\oint \sigma_{2,1}(A^T,L)\right)'=\oint\{(4-n)\phi\sigma_{2,1}(A^T,L)+\frac{n-2}{2}T_1(A)^n_j\phi^j-\sigma_2(L)\phi_n\}. \tag{31}$$

For $n>4$, using (27) we have $\left(\oint\sigma_3(L)\right)'=\oint\{(4-n)\sigma_3(L)\phi+(n-3)\sigma_2(L)\phi_n\}$. Recall that $\mathcal{B}^2=\frac{2}{n-2}\sigma_{2,1}(A^T,L)+\frac{2}{(n-2)(n-3)}\sigma_3(L)$. Hence, we obtain $\left(\oint\mathcal{B}^2\right)'=(4-n)\oint\mathcal{B}^2\phi+\oint T_1(A)^n_j\phi^j$. Going back to (28), we finally arrive at

$$\left(\int\sigma_2(A)dV+\oint\mathcal{B}^2d\Sigma-\Lambda\int dV\right)'=(4-n)\left(\int\sigma_k\phi+\oint\mathcal{B}^2\phi\right)-\Lambda\int n\phi$$

for constant Λ. Since $n-4\neq0$, critical points of $\mathcal{F}_2$ restricted on $\mathcal{M}$ satisfy $\sigma_2=constant$ in M and $\mathcal{B}^2=0$ in ∂M.

For $n=3$, note that $\frac{1}{3}h^3-\frac{1}{2}h|L|^2=-\frac{1}{6}\sigma_1(L)^3+\sigma_1(L)\sigma_2(L)$. Then by (27), we have

$$\left(\oint-\frac{1}{6}\sigma_1(L)^3+\sigma_1(L)\sigma_2(L)\right)' \;=\; \oint\{(-\frac{1}{6}\sigma_1(L)^3+\sigma_1(L)\sigma_2(L))\phi+2\sigma_2(L)\phi_n\}.$$

Recall that $\mathcal{B}^2=2\sigma_{2,1}(A^T,L)+\frac{1}{3}h^3-\frac{1}{2}h|L|^2$. Hence, we obtain $\left(\oint\mathcal{B}^2\right)'=\oint\mathcal{B}^2\phi+\oint T_1(A)^n_j\phi^j$. Now the rest of proof is the same as $n>4$ case.

(b) By (25),

$$\left(\oint\sigma_{2k-i-1,i}(A^T,L)\right)'=\oint\{(2k-n)\phi\sigma_{2k-i-1,i}(A^T,L)$$
$$+\frac{i}{2k-i-1}T_{2k-i-2,i-1}(A^T,L)^\alpha_\beta\phi^\beta_\alpha+\frac{2k-2i-1}{2k-i-1}(n-2k+i+1)\sigma_{2k-i-2,i}(A^T,L)\phi_n\}. \tag{32}$$

For the second term in the last integral, applying integration by parts we have

$$
\begin{aligned}
&\oint T_{2k-i-2,i-1}(A^T,L)^\alpha_\beta \phi^\beta_\alpha = \oint\{-T_{2k-i-2,i-1}(A^T,L)^\alpha_{\beta,\tilde\alpha}\phi^\beta - T_{2k-i-2,i-1}(A^T,L)^\alpha_\beta L^\beta_\alpha \phi_n\}\\
&= \oint\{-\frac{(n-2k+i+1)(2k-2i-1)}{2k-i-2}T_{2k-i-3,i-1}(A^T,L)^\alpha_\beta A^n_\alpha \phi^\beta\\
&+ (i-1)T_{2k-i-2,i-2}(A^T,L)^\alpha_\beta A^n_\alpha\phi^\beta - (2k-i-1)\sigma_{2k-i-1,i-1}(A^T,L)\phi_n\}, \qquad (33)
\end{aligned}
$$

where in the last equality we use Lemma 11(b) and Lemma 4(a).

Now recall that $\mathcal{B}^k = \sum_{i=0}^{k-1} C_1(n,k,i)\sigma_{2k-i-1,i}$. Combining (32) and (33) gives

$$
\left(\oint \mathcal{B}^k\right)' = (2k-n)\oint \mathcal{B}^k\phi + \oint I * A^n_\alpha\phi^\beta + \oint II * \phi_n,
$$

where
$I = \sum_{i=0}^{k-1} C_1(n,k,i)[-\frac{(n-2k+i+1)(2k-2i-1)i}{(2k-i-1)(2k-i-2)}T_{2k-i-3,i-1}(A^T,L)^\alpha_\beta + \frac{i(i-1)}{2k-i-1}T_{2k-i-2,i-2}(A^T,L)^\alpha_\beta]$ and $II = \sum_{i=0}^{k-1} C_1(n,k,i)[-i\sigma_{2k-i-1,i-1}(A^T,L) + \frac{(2k-2i-1)(n-2k+i+1)}{2k-i-1}\sigma_{2k-i-2,i}(A^T,L)]$. By definition, we have $C_1 = \frac{(2k-i-1)!(n-2k+i)!}{(n-k)!(2k-2i-1)!!\,i!}$. Straightforward computations yield

$$
\begin{aligned}
I &= \sum_{i=1}^{k-1} -\frac{(2k-i-3)!(n-2k+i+1)!}{(n-k)!(2k-2i-3)!!(i-1)!}T_{2k-i-3,i-1}(A^T,L)^\alpha_\beta\\
&+ \sum_{i=2}^{k-1}\frac{(2k-i-2)!(n-2k+i)!}{(n-k)!(2k-2i-1)!!(i-2)!}T_{2k-i-2,i-2}(A^T,L)^\alpha_\beta = -T_{k-2}(A^T)^\alpha_\beta,
\end{aligned}
$$

where the terms cancel out except the $i=k-1$ term in the first summation. For II,

$$
\begin{aligned}
II &= \sum_{i=1}^{k-1} -\frac{(2k-i-1)!(n-2k+i)!}{(n-k)!(2k-2i-1)!!(i-1)!}\sigma_{2k-i-1,i-1}(A^T,L)\\
&+ \sum_{i=0}^{k-1}\frac{(2k-i-2)!(n-2k+i+1)!}{(n-k)!(2k-2i-3)!!\,i!}\sigma_{2k-i-2,i}(A^T,L) = \sigma_{k-1}(A^T),
\end{aligned}
$$

where all terms are cancelled except the $i=k-1$ term in the second summation.

Finally, using Lemma 3 (c) and (d) we obtain

$$
\begin{aligned}
\left(\oint \mathcal{B}^k\right)' &= (2k-n)\oint \mathcal{B}^k\phi + \oint -T_{k-2}(A^T)^\alpha_\beta A^n_\alpha\phi^\beta + \oint \sigma_{k-1}(A^T)\phi_n\\
&= (2k-n)\oint \mathcal{B}^k\phi + \oint T_{k-1}(A)^n_j\phi^j.
\end{aligned}
$$

Hence, by (28) we arrive at

$$
\left(\int \sigma_k(A)dV + \oint \mathcal{B}^k d\Sigma - \Lambda\int dV\right)' = (2k-n)\left(\int \sigma_k\phi + \oint \mathcal{B}^k\phi\right) - \Lambda\int n\phi
$$

for constant Λ. Since $n-2k\neq 0$, this gives the result.

(c) First note that when the boundary is umbilic, by (24) we have $\mu' = \mu\phi + \phi_n$. Therefore, by (25) we have

$$\left(\oint \sigma_i(A^T)\mu^{2k-2i-1}\right)' = \oint \{(2k-n)\sigma_i(A^T)\mu^{2k-2i-1}\phi + T_{i-1}(A^T)^\alpha_\beta \phi^\beta_\alpha \mu^{2k-2i-1} + (2k-2i-1)\sigma_i(A^T)\mu^{2k-2i-2}\phi_n\}. \tag{34}$$

For the second term in the last integral, applying integration by parts we have

$$\begin{aligned}&\oint T_{i-1}(A^T)^\alpha_\beta \phi^\beta_\alpha \mu^{2k-2i-1} = \oint T_{i-1}(A^T)^\alpha_\beta (\phi^{\tilde{\beta}}_{\tilde{\alpha}} - \mu g^\beta_\alpha \phi_n)\mu^{2k-2i-1}\\ &= \oint \{(n-i)T_{i-2}(A^T)^\alpha_\beta \phi^\beta \mu^{2k-2i}\mu_\alpha - (2k-2i-1)T_{i-1}(A^T)^\alpha_\beta \mu_\alpha \phi^\beta\\ &- (n-i)\sigma_{i-1}(A^T)\mu^{2k-2i}\phi_n\},\end{aligned} \tag{35}$$

where in the last equality we use Lemma 11(c) and Lemma 6(a).

Recall that $\mathcal{B}^k = \sum_{i=0}^{k-1} C_2(n,k,i)\sigma_i \mu^{2k-2i-1}$. Combining (34) and (35) gives

$$\left(\oint \mathcal{B}^k\right)' = (2k-n)\oint \mathcal{B}^k\phi + \oint I * \mu_\alpha \phi^\beta + \oint II * \phi_n,$$

where
$I = \sum_{i=0}^{k-1} C_2(n,k,i)[-(2k-2i-1)T_{i-1}(A^T)^\alpha_\beta \mu^{2k-2i-2} + (n-i)T_{i-2}(A^T)^\alpha_\beta \mu^{2k-2i}]$ and $II = \sum_{i=0}^{k-1} C_2(n,k,i)[-(n-i)\sigma_{i-1}(A^T)\mu^{2k-2i} + (2k-2i-1)\sigma_i(A^T)\mu^{2k-2i-2}]$. By definition, we have $C_2 = \frac{(n-i-1)!}{(n-k)!(2k-2i-1)!!}$. Straightforward computations yield

$$\begin{aligned} I &= \sum_{i=1}^{k-1} -\frac{(n-i-1)!}{(n-k)!(2k-2i-3)!!} T_{i-1}(A^T)^\alpha_\beta \mu^{2k-2i-2}\\ &+ \sum_{i=2}^{k-1} \frac{(n-i)!}{(n-k)!(2k-2i-1)!!} T_{i-2}(A^T)^\alpha_\beta \mu^{2k-2i} = -T_{k-2}(A^T)^\alpha_\beta,\end{aligned}$$

where all terms are cancelled except the $i=k-1$ term in the first summation. For II ,

$$\begin{aligned} II &= \sum_{i=1}^{k-1} -\frac{(n-i)!}{(n-k)!(2k-2i-1)!!}\sigma_{i-1}(A^T)\mu^{2k-2i}\\ &+ \sum_{i=0}^{k-1} \frac{(n-i-1)!}{(n-k)!(2k-2i-3)!!}\sigma_i(A^T)\mu^{2k-2i-2} = \sigma_{k-1}(A^T),\end{aligned}$$

where all terms are cancelled except the $i=k-1$ term in the second summation.

Noting that by Lemma 6, we have $A_\alpha^n = \mu_\alpha$. As a result, we obtain $\left(\oint \mathcal{B}^k\right)' = (2k-n)\oint \mathcal{B}^k\phi + \oint -T_{k-2}(A^T)_\beta^\alpha \mu_\alpha \phi^\beta + \oint \sigma_{k-1}(A^T)\phi_n$. By Lemma 3 (c) and (d), this gives $(\oint \mathcal{B}^k)' = (2k-n)\oint \mathcal{B}^k\phi + \oint T_{k-1}(A)_j^n \phi^j$. Hence, by (28) we finally arrive at

$$\left(\int \sigma_k(A)dV + \oint \mathcal{B}^k d\Sigma - \Lambda \int dV\right)' = (2k-n)\left(\int \sigma_k\phi + \oint \mathcal{B}^k\phi\right) - \Lambda\int n\phi$$

for constant Λ. $\square$

Proof of Corollary 1. Let $g_t = e^{-2u_t}g$ be a conformal variation of g such that $u_0 = 0$ and $u_t'|_0 = \phi$. Since $\mathcal{L}(g_t) = e^{(2k-1)u_t}\mathcal{L}(g)$, we have $\mathcal{L}' = (2k-1)\phi\mathcal{L}$. Therefore, $\left(\oint \mathcal{L}d\Sigma\right)' = (2k-n)\oint \mathcal{L}d\Sigma$. Combining the above formula with the results of Theorem 3 gives $\left(\int \sigma_k(A)dV + \oint(\mathcal{B}^k + \mathcal{L})d\Sigma - \Lambda\int dV\right)' = (2k-n)(\int \sigma_k\phi + \oint(\mathcal{B}^k + \mathcal{L})\phi) - \Lambda\int n\phi$. $\square$

4 Conformal Invariants $\mathcal{Y}_k$

In this section, we first show that $\mathcal{F}_{\frac{n}{2}}$ is a conformal invariant and then we prove Theorem 4. Let $\mathcal{L}_4(g) = -2\sigma_1(A^T)h - 2(n-3)A_{\alpha\beta}L^{\alpha\beta} + 2R^\gamma_{\alpha\gamma\beta}L^{\alpha\beta}$, which satisfies $\mathcal{L}_4(\hat{g}) = e^{3u}\mathcal{L}_4(g)$; see [2].

Proposition 3. *Let (M,g) be a compact manifold of dimension $n \geq 3$ with boundary.*
(a) When $n = 4$, then $\mathcal{B}^2 = \frac{1}{2}\mathcal{B} + \frac{1}{4}\mathcal{L}_4$. Therefore, $\mathcal{F}_2 = 2\pi^2\chi(M,\partial M) - \frac{1}{16}\int|\mathcal{W}|^2 + \frac{1}{4}\oint\mathcal{L}_4$ is a conformal invariant.
(b) Suppose M is locally conformally flat. When $n = 2k$, then $\mathcal{F}_{\frac{n}{2}} = \frac{(2\pi)^{\frac{n}{2}}}{(\frac{n}{2})!}\chi(M,\partial M)$.

Proof of Proposition 3. (a) By Lemma 4 (a), we have $\mathcal{B}^2 = \sigma_{2,1}(A^T,L) + \sigma_{3,0}(A^T,L) = \frac{1}{2}\sigma_1(A^T)h - \frac{1}{2}L_\beta^\alpha A_\alpha^\beta + \frac{1}{3}trL^3 + \frac{1}{6}h^3 - \frac{1}{2}h|L|^2$, which is equal to $\frac{1}{2}\mathcal{B} + \frac{1}{4}\mathcal{L}_4$ by direct computations. Since $\mathcal{W}$ and $\mathcal{L}_4$ are local conformal invariants, $\mathcal{F}_2$ is then a conformal invariant.

(b) Recall the Gauss-Bonnet formulas $(4\pi)^{\frac{n}{2}}\chi(M,\partial M) = \int E_n dV + \oint\sum_i Q_{i,n}d\Sigma$, where $E_n = (2^{\frac{n}{2}}(\frac{n}{2})!)^{-1}\sum\begin{pmatrix} i_1\cdots i_n \\ j_1\cdots j_n\end{pmatrix} R_{i_1i_2}{}^{j_1j_2}\cdots R_{i_{n-1}i_n}{}^{j_{n-1}j_n}$ and
$Q_{i,n} = \frac{2^{\frac{n}{2}-2i}}{i!(n-1-2i)!!}\sum\begin{pmatrix}\alpha_1\cdots\alpha_{n-1}\\ \beta_1\cdots\beta_{n-1}\end{pmatrix} R_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\cdots R_{\alpha_{2i-1}\alpha_{2i}}{}^{\beta_{2i-1}\beta_{2i}} L^{\beta_{2i+1}}_{\alpha_{2i+1}}\cdots L^{\beta_{n-1}}_{\alpha_{n-1}}$. When the manifold is locally conformally flat, by the curvature decomposition $R_{ijkl} = A_{ik}g_{jl} + A_{jl}g_{ik} - A_{il}g_{jk} - A_{jk}g_{il}$. It has been shown in [28] that $E_n = 2^{\frac{n}{2}}(\frac{n}{2})!\sigma_{\frac{n}{2}}(A)$. We only need to compute $Q_{i,n}$.

$$\begin{aligned} Q_{i,n} &= \frac{2^{\frac{n}{2}-2i}}{i!(n-1-2i)!!}\sum\begin{pmatrix}\alpha_1\cdots\alpha_{n-1}\\ \beta_1\cdots\beta_{n-1}\end{pmatrix} 2^i(A^{\beta_1}_{\alpha_1}g^{\beta_1}_{\alpha_1} + A^{\beta_2}_{\alpha_2}g^{\beta_1}_{\alpha_1})\cdots \\ &\quad (A^{\beta_{2i-1}}_{\alpha_{2i-1}}g^{\beta_{2i}}_{\alpha_{2i}} + A^{\beta_{2i}}_{\alpha_{2i}}g^{\beta_{2i-1}}_{\alpha_{2i-1}})L^{\beta_{2i+1}}_{\alpha_{2i+1}}\cdots L^{\beta_{n-1}}_{\alpha_{n-1}} \\ &= \frac{2^{\frac{n}{2}-2i}}{i!(n-1-2i)!!}\sum\begin{pmatrix}\alpha_1\cdots\alpha_i\,\alpha_{2i+1}\cdots\alpha_{n-1}\\ \beta_1\cdots\beta_i\,\beta_{2i+1}\cdots\beta_{n-1}\end{pmatrix} i!2^{2i}A^{\beta_1}_{\alpha_1}\cdots A^{\beta_i}_{\alpha_i}L^{\beta_{2i+1}}_{\alpha_{2i+1}}\cdots L^{\beta_{n-1}}_{\alpha_{n-1}} \\ &= \frac{2^{\frac{n}{2}}(n-1-i)!}{(n-1-2i)!!}\sigma_{n-1-i,i}(A^T,L) = 2^{\frac{n}{2}}(\frac{n}{2})!C_1(n,\frac{n}{2},i). \end{aligned}$$

$\square$

Proof of Theorem 4. We will show that there exists a conformal metric $\hat{g}$ such that $A_{\hat{g}} \in \Gamma_k^+$ and the boundary is totally geodesic. Then by the result in [7], we can find a conformal metric $\tilde{g}$ such that $\sigma_k(A_{\tilde{g}}) = 1$ and the boundary is totally geodesic.

Let the background metric g be a Yamabe metric such that $R = constant > 0$ and the boundary is totally geodesic. We prove inductively that we can find $\hat{g}$ such that $A_{\hat{g}} \in \Gamma_m^+$ for $m \leq k$. Suppose g satisfies $A_g \in \Gamma_{m-1}^+$ and the boundary is totally geodesic. Define $A_{m-1}^t = A + \frac{1-t}{2}\sigma_{m-1}^{\frac{1}{m-1}}(A)g$. Under the conformal change $\hat{g} = e^{-2u}g$, the tensor $\hat{A}_{m-1}^t$ satisfies $\hat{A}_{m-1}^t = \hat{A} + \frac{1-t}{2}\sigma_{m-1}^{\frac{1}{m-1}}(g^{-1}\hat{A})g$, where $\hat{A} = \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A$. Since $\sigma_{m-1}(A)$ is positive, we choose a large number Θ such that $A_{m-1}^{-\Theta}$ is positive definite. Let $f(x) = \sigma_m^{\frac{1}{m}}(A_{m-1}^{-\Theta}) > 0$. Consider the following path of equations for $-\Theta \leq t \leq 1:$

$$\begin{cases} \sigma_m^{\frac{1}{m}}(g^{-1}\hat{A}_{m-1}^t) = f(x)e^{2u} & in\, M \\ \frac{\partial u}{\partial n} = 0 & on\, \partial M, \end{cases} \tag{36}$$

where $\hat{A} \in \Gamma_m^t = \{\lambda : \lambda \in \Gamma_{m-1}^+, \lambda + \frac{1-t}{2}\sigma_{m-1}^{\frac{1}{m-1}}(\lambda)e \in \Gamma_m^+\}$. Note that if $\lambda + \frac{1-t}{2}\sigma_{m-1}^{\frac{1}{m-1}}(\lambda)e \in \Gamma_m^+$, then we must have $\lambda \in \Gamma_{m-1}^+$ along the path because $\sigma_{m-1}(\lambda + \frac{1-t}{2}\sigma_{m-1}^{\frac{1}{m-1}}(\lambda)e)$ can not be zero.

Let $\mathcal{S} = \{t \in [-\Theta, 1] : \exists$ a solution $u \in C^{2,\alpha}(M)$ to (36) with $\hat{A} \in \Gamma_m^t\}$. At $t = -\Theta$, we have $u \equiv 0$ is a solution and $A_{m-1}^{-\Theta} \in \Gamma_m^+$. Consider the linearized operator $\mathcal{P}^t:$

Lemma 12. *The linearized operator $\mathcal{P}^t : C^{2,\alpha}(M) \cap \{\frac{\partial u}{\partial n}|_{\partial M} = 0\} \to C^\alpha(M)$ is invertible.*

Proof. Let $F^t = \sigma_m(g^{-1}\hat{A}_{m-1}^t) - f^m e^{2mu}$ and u_s be a variation of u such that $u' = \phi$ at $s = 0$. Then

$$\begin{aligned} \mathcal{P}^t &= (F^t)'|_{s=0} = T_{m-1}(g^{-1}\hat{A}_{m-1}^t)^{ij}(g^{-1}\hat{A}_{m-1}^t)'_{ij} - 2mf^m e^{2mu}\phi \\ &= [T_{m-1}(g^{-1}\hat{A}_{m-1}^t)^{ij} + \frac{1-t}{2}\sigma_{m-1}^{-\frac{m-2}{m-1}}(g^{-1}\hat{A})tr_g T_{m-1}(g^{-1}\hat{A}_{m-1}^t)T_{m-2}(g^{-1}\hat{A})^{ij}]\phi_{ij} \\ &+ \text{1st derivatives in}\, \phi - 2mf^m e^{2mu}\phi. \end{aligned}$$

Since the terms in the parenthesis are positive, the linearized operator is invertible. $\square$

The above lemma and the implicit function theorem imply that $\mathcal{S}$ is open. To complete the proof, it remains to establish a priori estimates for solutions to (36).

(1) C^0 estimates.

Since $\frac{\partial u}{\partial n} = 0$, at the maximal point x_0 of u, we have $|\nabla u| = 0$ and $\nabla^2 u(x_0)$ is negative semi-definite, no matter x_0 being in the interior or at the boundary. Hence, $f(x_0)e^{2u(x_0)} = \sigma_m^{\frac{1}{m}}(g^{-1}\hat{A}_{m-1}^t) \leq \sigma_m^{\frac{1}{m}}(\frac{1-t}{2}\sigma_{m-1}^{\frac{1}{m-1}}(A)g + A) \leq C$, where in the inequality we use $t \leq 1$. Therefore, u is upper bounded.

Now by [6] and Theorem 5 (a), we have $|\nabla u| < C$. Thus, $\sup_M u \leq \inf_M u + C$. Integrating the equation,

$$\begin{aligned}\int f^m e^{4mu} dV_{\hat g} &= \int e^{2mu}\sigma_m(g^{-1}\hat A^t_{m-1})dV_{\hat g} = \int \sigma_m(\hat g^{-1}\hat A^t_{m-1})dV_{\hat g}\\ &= \int \sum_{i=0}^m \binom{n-i}{m-i}(\tfrac{1-t}{2})^{m-i}\sigma_i(\hat g^{-1}\hat A)\sigma_{m-1}^{\frac{m-i}{m-1}}(\hat g^{-1}\hat A)dV_{\hat g} \geq \int \sigma_m(\hat g^{-1}\hat A)dV_{\hat g},\end{aligned}$$

where we drop the terms for $i = 0, \cdots, m-1$, which are nonnegative. Since the boundary is totally geodesic, we have $\mathcal{B}^k = 0$. Therefore,

$$0 < \mathcal{Y}_m \leq \frac{\int f^m e^{4mu} dV_{\hat g}}{(\int dV_{\hat g})^{\frac{n-2m}{n}}} \leq C(\sup e^{4mu})(\int dV_{\hat g})^{\frac{2m}{n}} \leq Ce^{4m\sup u}e^{-2m\inf u}.$$

Since $\sup_M u \leq \inf_M u + C$, we then have $0 < \mathcal{Y}_m \leq Ce^{2m\sup u + C}$.

(2) C^∞ estimates

By [6] and Theorem 5 (a), we get interior and boundary C^2 estimates, respectively. Higher order regularity follows the same way as in (3) in the proof of Proposition 1. $\square$

5 Proofs of Theorem 5 and Corollary 2

Proof of Theorem 5. Let $W = \nabla^2 u + du\otimes du - \frac{1}{2}|\nabla u|^2 g + S(x)$. The condition $\Gamma_1^+ \subset \Gamma$ gives $0 < tr_g\,\hat A = \Delta u - \frac{(n-2)}{2}|\nabla u|^2 + tr_g S(x)$. Thus, Δu has a lower bound and

$$|\nabla u|^2 < C(\Delta u + 1). \tag{37}$$

We first prove a lemma which will be used later to control the boundary behavior of u.

Lemma 13. *Let W be defined as above. Under the same conditions as in Theorem 5, we have*
(a) $W_{n\alpha} = 0$ on ∂M and hence $F_{\alpha n} = 0$ on ∂M;
(b) $W_{\alpha\beta,n} - 2\mu W_{\alpha\beta} \leq -\hat\mu e^{-u}(W_{\alpha\beta} + W_{nn}g_{\alpha\beta})$.

Proof. (a) By (13) and (T0),

$$W_{\alpha n} = u_{\alpha n} + u_n u_\alpha + S_{\alpha n} = -\mu_\alpha + \mu u_\alpha - \hat\mu u_\alpha e^{-u} + (-\mu + \hat\mu e^{-u})u_\alpha + S_{\alpha n} = 0.$$

To prove $F_{\alpha n} = 0$, since F is a function of σ_i, we only need to show that $\frac{\partial\sigma_i(W)}{\partial W_{\alpha n}} = (T_{i-1})_{\alpha n} = 0$ for all i. For $i = 1$, by definition $(T_1)_{\alpha n} = \sigma_1(W)g_{\alpha n} - W_{\alpha n} = 0$. For general i, notice the recursive relation $(T_i)_{\alpha n} = \sigma_i(W)g_{\alpha n} - (T_{i-1})_{\alpha j}W_{jn}$. Applying the induction hypothesis gives $(T_i)_{\alpha n} = -(T_{i-1})_{\alpha\beta}W_{\beta n} = 0$.

(b) By (13) and (14),

$$\begin{aligned} W_{\alpha\beta,n} &= u_{\alpha\beta n} + u_\alpha u_{\beta n} + u_{\alpha n}u_\beta - u_l u_{ln} g_{\alpha\beta} + S_{\alpha\beta}\\ &= (2\mu - \hat\mu e^{-u})(u_{\alpha\beta} + u_\alpha u_\beta) - \mu_{\tilde\alpha\tilde\beta} + R_{n\beta\alpha n}(-\mu + \hat\mu e^{-u}) - \mu u_n^2 g_{\alpha\beta}\\ &\quad -(\mu - \hat\mu e^{-u})u_\gamma^2 g_{\alpha\beta} - \hat\mu e^{-u}u_{nn}g_{\alpha\beta} + S_{\alpha\beta,n}.\end{aligned}$$

Therefore,

$$\begin{aligned}W_{\alpha\beta,n} &= (2\mu-\hat{\mu}e^{-u})W_{\alpha\beta}-\hat{\mu}e^{-u}W_{nn}-(2\mu-\hat{\mu}e^{-u})S_{\alpha\beta}-\mu_{\tilde{\alpha}\tilde{\beta}}+R_{n\beta\alpha n}(-\mu+\hat{\mu}e^{-u})\\&\quad+\hat{\mu}e^{-u}S_{nn}g_{\alpha\beta}+S_{\alpha\beta,n}.\end{aligned}$$

Now by (T1) and (T2), we arrive at

$$\begin{aligned}W_{\alpha\beta,n} &\leq (2\mu-\hat{\mu}e^{-u})W_{\alpha\beta}-\hat{\mu}e^{-u}W_{nn}+\hat{\mu}e^{-u}S_{\alpha\beta}-R_{\beta n\alpha n}\hat{\mu}e^{-u}+\hat{\mu}e^{-u}S_{nn}g_{\alpha\beta}\\&\leq (2\mu-\hat{\mu}e^{-u})W_{\alpha\beta}-\hat{\mu}e^{-u}W_{nn},\end{aligned}$$

where the last inequality is by nonnegativity of $\hat{\mu}$. $\square$

We continue the proof of Theorem 5.

(1) We show that on the boundary u_{nnn} can be controlled from below by Δu. More specifically, we have $u_{nnn}\geq -L\Delta u+3\mu u_{nn}-C$ for some number L independent of points on the boundary.

At a boundary point, differentiating the equation on both sides in the normal direction, we get

$$(f(x,u))_n = F^{\alpha\beta}W_{\alpha\beta,n}+F^{nn}W_{nn,n},$$

where we have used $F^{\alpha n}=0$ by Lemma 13.

For case (a), by Lemma 13 again, $W_{\alpha\beta,n}-2\mu W_{\alpha\beta}\leq 0$. Thus,

$$\begin{aligned}(f(x,u))_n &\leq 2\mu F^{\alpha\beta}W_{\alpha\beta}+F^{nn}W_{nn,n}\\&= 2\mu F+F^{nn}(W_{nn,n}-2\mu W_{nn}) = 2\mu f(x,u)+F^{nn}(W_{nn,n}-2\mu W_{nn}),\end{aligned}\tag{38}$$

where the first equality holds by Lemma 1 (a). By (13) and the boundary condition,

$$\begin{aligned}W_{nn,n}-2\mu W_{nn} &= u_{nnn}+2u_nu_{nn}-u_lu_{ln}+S_{nn,n}-2\mu(u_{nn}+u_n^2-\frac{1}{2}|\nabla u|^2+S_{nn})\\&= u_{nnn}-3\mu u_{nn}+u_\alpha\mu_\alpha+S_{nn,n}-\mu^3-2\mu S_{nn}.\end{aligned}$$

Returning to (38), we use the conditions $|\nabla_x f|\leq \Lambda f$ and $|f_z|\leq\Lambda f$ to get

$$-Cf\leq f_{x_n}+f_zu_n-2\mu f\leq F^{nn}(W_{nn,n}-2\mu W_{nn})\leq F^{nn}(u_{nnn}-3\mu u_{nn}+u_\alpha\mu_\alpha+C).$$

On the other hand, by condition (S3) we have $F^{nn}\geq\epsilon\frac{F}{\sigma_1}\geq\epsilon\frac{f(x,u)}{\Delta u+C}$. Hence, there is a positive number L such that

$$u_{nnn}\geq -L\Delta u+3\mu u_{nn}-u_\alpha\mu_\alpha-C\tag{39}$$

is true for every point on the boundary, where L and C depend on $n,\epsilon,\mu,c_{\sup}$ and Λ.

For case (b), by Lemma 13 (b) we get

$$\begin{aligned}(f(x,u))_n &\leq \sum_{\alpha,\beta}F^{\alpha\beta}(2\mu W_{\alpha\beta}-\hat{\mu}e^{-u}(W_{\alpha\beta}+W_{nn}g_{\alpha\beta}))+F^{nn}W_{nn,n}\\&= (2\mu-\hat{\mu}e^{-u})f(x,u)-\hat{\mu}e^{-u}\sum_\alpha F^{\alpha\alpha}W_{nn}+F^{nn}(W_{nn,n}-(2\mu-\hat{\mu}e^{-u})W_{nn}),\end{aligned}$$

where the equality holds by Lemma 1 (a). Using the conditions $|\nabla_x f| \le \Lambda f$ and $|f_z| \le \Lambda f$, the above formula becomes

$$-Cf \le f_{x_n} + f_z u_n - (2\mu - \hat{\mu} e^{-u}) f \le -\hat{\mu} e^{-u} \sum_{\alpha} F^{\alpha\alpha} W_{nn} + F^{nn}(W_{nn,n} - (2\mu - \hat{\mu} e^{-u}) W_{nn}),$$

where C depends on $\inf u$. Since $\hat{\mu}$ is positive, if $W_{nn} \ge 0$, then $-Cf \le F^{nn}(W_{nn,n} - (2\mu - \hat{\mu} e^{-u}) W_{nn})$. On the other hand, if $W_{nn} < 0$, by condition (A) we have $-Cf \le F^{nn}(W_{nn,n} - (2\mu + \rho\, \hat{\mu} e^{-u}) W_{nn})$, where we drop the term $F^{nn} \hat{\mu} e^{-u} W_{nn}$ since it is negative. Hence, in both cases we obtain

$$-Cf \le F^{nn}(W_{nn,n} - 2\mu W_{nn} + C|W_{nn}|). \tag{40}$$

Now by (13) and (14) and combined with a basic fact that if $\Gamma_2^+ \subset \Gamma$, then $|u_{ij}| \le C\Delta u$, we get

$$W_{nn,n} - 2\mu W_{nn} + C|W_{nn}| \le u_{nnn} + (-3\mu + \hat{\mu} e^{-u}) u_{nn} + C\Delta u + C.$$

Returning to (40), note that by condition (S3) we have $F^{nn} \ge \epsilon \frac{F}{\sigma_1} \ge \epsilon \frac{f(x,u)}{\Delta u + C}$. Hence, there is a positive number L such that

$$u_{nnn} \ge -L\Delta u + (3\mu - \hat{\mu} e^{-u}) u_{nn} - C \tag{41}$$

is true for every point on the boundary, where L and C depends on $n, \epsilon, \rho, \mu, \hat{\mu}, \inf u, c_{\sup}$ and Λ.

(2) We will show that Δu is bounded. The follow proof is for both cases (a) and (b), while the number C is understood as a constant depending on $n, r, \epsilon, \mu, c_{\sup}$ and Λ for case (a), and $n, r, \epsilon, \rho, \mu, \hat{\mu}, \inf u, c_{\sup}$ and Λ for case (b), respectively.

Define μ on the half ball in Fermi coordinates by $\mu(x', x_n) = \mu(x')$, where $x' = (x_1, \cdots, x_{n-1})$. Let $H = \eta(\Delta u + |\nabla u|^2 + n\mu\, u_n) e^{a\, x_n} = \eta K e^{a\, x_n}$ where a is some number chosen later. Denote $r^2 :\equiv \sum_i x_i^2$. Let $\eta(r)$ be a cutoff function such that $0 \le \eta \le 1$, $\eta = 1$ in $\overline{B}^+_{\frac{r}{2}}$ and $\eta = 0$ outside $\overline{B}^+_r$, and also $|\nabla \eta| < C\frac{\eta^{\frac{1}{2}}}{r}$ and $|\nabla^2 \eta| < \frac{C}{r^2}$. By (37), Δu is lower bounded. Without loss of generality, we may assume $r = 1$ and

$$K = \Delta u + |\nabla u|^2 + n\mu\, u_n \gg 1.$$

At a boundary point, since $\eta = \eta(r)$, we have $\eta_n = 0$. Differentiating H in the normal direction produces

$$H_n = \eta(K_n + aK) e^{a x_n} = \eta(u_{nnn} + u_{\alpha\alpha n} + (2u_n + n\mu) u_{nn} + 2u_\alpha u_{\alpha n} + aK) e^{a x_n}.$$

Using (13) and (14) gives

$$\begin{aligned} H_n \ \ge\ & \eta(u_{nnn} - \tilde{\Delta}\mu + (2\mu - \hat{\mu} e^{-u}) u_{\alpha\alpha} + (-\mu + 2\hat{\mu} e^{-u}) u_{nn} + (2\mu - \hat{\mu} e^{-u}) u_\alpha u_\alpha \\ & -(n-1)\mu_\alpha u_\alpha - \mu(n-1)(-\mu + \hat{\mu} e^{-u})^2 - R_{nn}(-\mu + \hat{\mu} e^{-u}) + aK - C) e^{a x_n} \\ \ge\ & \eta(u_{nnn} - (n-1)\mu_\alpha u_\alpha + (2\mu - \mu e^{-u}) K + (-3\mu + \mu e^{-u}) u_{nn} + aK - C) e^{a x_n}. \end{aligned}$$

By (37) and the inequalities (39) and (41) for cases (a) and (b), respectively, we obtain

$$H_n \geq \eta(-L\Delta u + (2\mu - \hat{\mu}e^{-u})K - (n-1)\mu_\alpha u_\alpha - C + aK)e^{ax_n} > 0$$

for $a > L - 2\mu + \hat{\mu}\sup e^{-u} + 1$. Thus, H increases toward the interior and the maximum of H must happen at some point x_0 in the interior.

Now we know the maximal point x_0 is in the interior. Thus, at x_0 we have

$$H_i = \eta_i(Ke^{ax_n}) + \eta e^{ax_n}(K_i + aK\delta_{in}) = 0, \tag{42}$$

and

$$H_{ij} = \eta_{ij}(Ke^{ax_n}) + \eta_i(Ke^{ax_n})_j + \eta_j(Ke^{ax_n})_i + \eta(Ke^{ax_n})_{ij},$$

is negative semi-definite. Using (42), the above formula becomes

$$H_{ij} = (\eta_{ij} - 2\eta^{-1}\eta_i\eta_j)Ke^{ax_n} + \eta e^{ax_n}(K_{ij} + aK_i\delta_{jn} + aK_j\delta_{in} + a^2K\delta_{in}\delta_{jn}).$$

Using the positivity of F^{ij}, and (42) to replace K_i and K_j, we get

$$0 \geq F^{ij}H_{ij}e^{-ax_n} = F^{ij}((\eta_{ij} - 2\eta^{-1}\eta_i\eta_j)K + \eta(K_{ij} - a\frac{\eta_i}{\eta}K\delta_{jn} - a\frac{\eta_j}{\eta}K\delta_{in} - a^2K\delta_{in}\delta_{jn})).$$

Therefore,

$$0 \geq \eta F^{ij}K_{ij} - C\sum_i F^{ii}K, \tag{43}$$

where we use conditions on η.

By direct computations, we have

$$F^{ij}K_{ij} = F^{ij}(u_{llij} + 2u_{li}u_{lj} + 2u_lu_{lij} + n\mu_{ij}u_n + n\mu_iu_{nj} + n\mu_ju_{ni} + n\mu u_{nij}).$$

Changing the order of the covariant differentiations and using (37) give

$$\begin{aligned} F^{ij}K_{ij} &\geq F^{ij}u_{ijll} + F^{ij}(2u_{li}u_{lj} + 2u_lu_{ijl} + n\mu u_{ijn}) - C\sum_i F^{ii}(1 + |\nabla^2 u|) \\ &= I + II - C\sum_i F^{ii}(1 + |\nabla^2 u|). \end{aligned}$$

For I, notice that

$$W_{ij,ll} = u_{ijll} + 2u_{il}u_{jl} + u_iu_{jll} + u_ju_{ill} - (u_ku_{kll} + u_{kl}^2)g_{ij} + S_{ij,ll}.$$

Then

$$I = F^{ij}(W_{ij,ll} - 2u_{li}u_{lj} - 2u_{ill}u_j + (u_{lk}^2 + u_ku_{kll})g_{ij} - S_{ij,ll}),$$

where $F^{ij}(u_iu_{jll}) = F^{ij}(u_ju_{ill})$ because F^{ij} is symmetric. Changing the order of differentiations again yields

$$I \geq F^{ij}W_{ij,ll} + F^{ij}(-2u_{li}u_{lj} - 2u_{lli}u_j + (u_{lk}^2 + u_ku_{llk})g_{ij}) - C\sum_i F^{ii}(1 + |\nabla^2 u|).$$

Now replace u_{lli} and u_{llk} by (42) to get

$$\begin{aligned} I &\geq F^{ij}W_{ij,ll} + F^{ij}(-2u_{li}u_{lj} - 2u_j(-2u_lu_{li} - n\mu u_{ni} - n\mu_i u_n - \frac{\eta_i}{\eta}K - aK\delta_{in}) \\ &+(|\nabla^2 u|^2 + u_k(-2u_lu_{lk} - n\mu u_{nk} - n\mu_k u_n - \frac{\eta_k}{\eta}K - aK\delta_{kn}))g_{ij}) \\ &-C\sum_i F^{ii}(1+|\nabla^2 u|). \end{aligned}$$

By (37) and the conditions on η, we have

$$\begin{aligned} I &\geq F^{ij}W_{ij,ll} + F^{ij}(-2u_{li}u_{lj} + 4u_ju_lu_{li} + (|\nabla^2 u|^2 - 2u_ku_lu_{lk})g_{ij}) \\ &-C\sum_i F^{ii}\eta^{-\frac{1}{2}}(1+|\nabla^2 u|^{\frac{3}{2}}). \end{aligned}$$

For II, we use the formula

$$W_{ij,l} = u_{ijl} + u_iu_{jl} + u_ju_{il} - u_ku_{kl}g_{ij} + S_{ij,l}$$

to obtain

$$\begin{aligned} II &= F^{ij}(2u_{li}u_{lj} + 2u_l(W_{ij,l} - 2u_iu_{jl} + u_ku_{kl}g_{ij} - S_{ij,l}) \\ &+n\mu(W_{ij,n} - 2u_iu_{jn} + u_ku_{kn}g_{ij} - S_{ij,n})) \\ &\geq F^{ij}(2u_{li}u_{lj} + 2u_lW_{ij,l} - 4u_iu_{jl}u_j + 2u_ku_{kl}u_lg_{ij} + n\mu W_{ij,n}) \\ &-C\sum_i F^{ii}(1+|\nabla^2 u|^{\frac{3}{2}}). \end{aligned}$$

Combining I and II together, we find that

$$\begin{aligned} F^{ij}K_{ij} &\geq F^{ij}W_{ij,ll} + F^{ij}(-2u_{li}u_{lj} + 4u_ju_lu_{li} + (|\nabla^2 u|^2 - 2u_ku_lu_{lk})g_{ij}) \\ &+F^{ij}(2u_{li}u_{lj} + 2u_lW_{ij,l} - 4u_iu_{jl}u_j + 2u_ku_{kl}u_lg_{ij} + n\mu W_{ij,n}) \\ &-C\sum_i F^{ii}\eta^{-\frac{1}{2}}(1+|\nabla^2 u|^{\frac{3}{2}}). \end{aligned}$$

Here is the key step of the proof. Three terms from I cancel out three terms from II. Thus, after the cancellations we arrive at

$$\begin{aligned} F^{ij}K_{ij} &\geq F^{ij}W_{ij,ll} + F^{ij}|\nabla^2 u|^2 g_{ij} + F^{ij}(2u_lW_{ij,l} + n\mu W_{ij,n}) \\ &-C\sum_i F^{ii}\eta^{-\frac{1}{2}}(1+|\nabla^2 u|^{\frac{3}{2}}). \end{aligned}$$

Now returning to (43), applying η on both sides produces

$$\begin{aligned} 0 &\geq \eta^2F^{ij}W_{ij,ll} + \eta^2F^{ij}|\nabla^2 u|^2 g_{ij} + \eta^2F^{ij}(2u_lW_{ij,l} + n\mu W_{ij,n}) \\ &-C\sum_i F^{ii}(1+\eta^{\frac{3}{2}}|\nabla^2 u|^{\frac{3}{2}}). \end{aligned}$$

By the concavity of F, we have $F^{ij}W_{ij,ll} \geq (f(x,u))_{ll}$. Hence,

$$\begin{aligned}
0 &\geq \eta^2 \sum_i F^{ii}|\nabla^2 u|^2 + \eta^2 (f(x,u))_{ll} + 2\eta^2 u_l (f(x,u))_l + n\mu\eta^2 (f(x,u))_n \\
&\quad -C \sum_i F^{ii}(1 + \eta^{\frac{3}{2}}|\nabla^2 u|^{\frac{3}{2}}) \\
&\geq \sum_i F^{ii}(\eta^2|\nabla^2 u|^2 - C - C\eta|\nabla^2 u| - C\eta^{\frac{3}{2}}|\nabla^2 u|^{\frac{3}{2}}).
\end{aligned}$$

This gives $(\eta|\nabla^2 u|)(x_0) \leq C$. Hence, for $x \in \overline{B}^+_{\frac{r}{2}}$, we have that $H = (\Delta u + |\nabla u|^2 + n\mu\, u_n)e^{a\, x_n}$ is bounded. Thus, Δu is bounded. By (37), $|\nabla u|$ is also bounded.

(3) To get the Hessian bounds, for case (b) it follows immediately by the fact that if $\Gamma_2^+ \subset \Gamma$, then $|u_{ij}| \leq C\Delta u$. As for case (a), note that from (2) above, we have $\eta\Delta u < C$ and $\eta|\nabla u|^2 < C$. Consider the maximum of $\eta(\nabla^2 u + du \otimes du + \mu u_n g)e^{ax_n}$ over the set $(x,\xi) \in (B_1^+, \mathbb{S}^n)$. We will show that at the maximum, x can not belong to the boundary. If ξ is in the tangential direction, without loss of generality, we can assume ξ is in e_1 direction. By formulas (13) and (14), we obtain

$$\begin{aligned}
&(\eta(u_{11} + u_1^2 + \mu u_n)e^{ax_n})_n \\
= \;& \eta e^{ax_n}((2\mu + a)(u_{11} + u_1^2 + \mu u_n) + \mu^3 - \mu_{\tilde{1}\tilde{1}} - \mu_\alpha u_\alpha - R_{n11n}\mu) \\
\geq \;& \eta e^{ax_n}((2\mu + a)(u_{11} + u_1^2 + \mu u_n) - \mu_\alpha u_\alpha - C) > 0
\end{aligned}$$

for $a > -2\mu + 1$. If ξ is in the normal direction, we first have that $\Delta u \leq n(u_{nn} + \mu^2) \leq nu_{nn} + C$. By (39) and (37), we obtain

$$\begin{aligned}
(\eta(u_{nn} + u_n^2 + \mu u_n)e^{ax_n})_n &= \eta(u_{nnn} - \mu u_{nn} + au_{nn})e^{ax_n} \\
&\geq \eta e^{ax_n}(-L\Delta u + 2\mu u_{nn} + au_{nn} - C_0\Delta u - C) \\
&\geq \eta e^{ax_n}(-n(L + C_0)u_{nn} + 2\mu u_{nn} + au_{nn} - C) > 0
\end{aligned}$$

for $a > n(L + C_0) - 2\mu + 1$. Thus, we conclude that at the maximum, x must be in the interior. We then perform similar computations as before using the inequality $\eta|\nabla u|^2 < C$ to get the Hessian bounds. We omit the details here. □

Proof of Corollary 2. It has been proved in Section 1 that A_g satisfies (T0)-(T2). We only need to verify the dependence of Λ and $C_{\sup}$ in Theorem 5.

Let $\tilde{f}(x,z) = f(x)e^{-2z}$ and $\Lambda = \frac{\|f\|_{C^1}}{\inf f} + 2$. Then

$$|\nabla_x \tilde{f}| \leq |\nabla f|e^{-2z} \leq \Lambda(fe^{-2z}) = \Lambda\tilde{f} \quad \text{and} \quad |\tilde{f}_z| = 2fe^{-2z} \leq \Lambda\tilde{f}.$$

For $c_{\sup}$, it is easy to see that $c_{\sup} \leq C\|f\|_{C^2} \sup e^{-2u} = C(\|f\|_{C^2}, \inf u)$. □

6 Proof of Theorem 6

In this section, we prove Theorem 6.

Proof. (a) Let $\hat{A} = \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g$ and $W = \hat{A} + S$. Recall that $T_1(W) = (tr_g W)g - W$ is the first Newton tensor and $F^{ij} = \frac{1}{2F}(T_1)_{ij}$, where $F = \sigma_2^{\frac{1}{2}}$. Since F^{ij} is positive, we have

$$T_1(W)_{nn} = u_{\alpha\alpha} - \frac{n-3}{2}|\nabla u|^2 - u_n^2 + T_1(A)_{nn} + trS - S_{nn} > 0.$$

Thus,

$$|\nabla u|^2 < C(1 + u_{\alpha\alpha}). \tag{44}$$

We will show that $u_{\alpha\alpha}$ and hence $|\nabla u|^2$ are bounded. Define μ on the half ball in Fermi coordinates by $\mu(x', x_n) = \mu(x')$, where $x' = (x_1, \cdots, x_{n-1})$. Let $G = \eta(u_{\alpha\alpha} + u_\alpha u_\alpha + (n-1)\mu u_n)e^{ax_n} = \eta E e^{ax_n}$, where a is some number chosen later. Denote $r^2 :\equiv \sum_i x_i^2$. Let $\eta(r)$ be a cutoff function as in the proof of Theorem 5 (2). Without loss of generality, we may assume $r = 1$ and

$$E = u_{\alpha\alpha} + u_\alpha u_\alpha + (n-1)\mu u_n \gg 1.$$

Therefore, by (44) we get $u_{\alpha\alpha} \gg 1$. Hence, we also have $u_\alpha u_\alpha < E$ on the boundary.

At a boundary point, since $\eta = \eta(r)$, we have $\eta_n = 0$. Differentiating G in the normal direction produces

$$G_n = \eta(E_n + aE)e^{ax_n} = \eta(u_{\alpha\alpha n} + 2u_\alpha u_{\alpha n} + (n-1)\mu u_{nn} + aE)e^{ax_n}.$$

Using (13) and (14) gives

$$\begin{aligned} G_n &= \eta(2\mu u_{\alpha\alpha} + 2\mu u_\alpha u_\alpha - (n-1)\mu^3 - \tilde{\Delta}\mu - (n-1)\mu_\alpha u_\alpha + \mu R_{nn} + aE)e^{ax_n} \\ &\geq \eta(2\mu E - (n-1)\mu_\alpha u_\alpha + aE - C)e^{ax_n}. \end{aligned}$$

By (44), we obtain

$$G_n \geq \eta((2\mu + a)E - (n-1)\mu_\alpha u_\alpha - C)e^{ax_n} > 0$$

for $a > -2\mu + 1$. Hence, the maximum of G must happen in the interior.

Now we know the maximal point x_0 is in the interior. Thus, at x_0 we have

$$G_i = \eta_i(Ee^{ax_n}) + \eta e^{ax_n}(E_i + aE\delta_{in}) = 0, \tag{45}$$

and

$$G_{ij} = \eta_{ij}(Ee^{ax_n}) + \eta_i(Ee^{ax_n})_j + \eta_j(Ee^{ax_n})_i + \eta(Ee^{ax_n})_{ij}$$

is negative semi-definite. Using (45), the above formula becomes

$$G_{ij} = (\eta_{ij} - 2\eta^{-1}\eta_i\eta_j)Ee^{ax_n} + \eta e^{ax_n}(E_{ij} + aE_i\delta_{jn} + aE_j\delta_{in} + a^2 E\delta_{in}\delta_{jn}).$$

Moreover, direct computations show

$$\begin{aligned} E_{ij} &= u_{\alpha\alpha ij} + 2u_{\alpha i}u_{\alpha j} + 2u_\alpha u_{\alpha ij} + (n-1)\mu_{ij}u_n + (n-1)\mu_i u_{nj} \\ &\quad + (n-1)\mu_j u_{ni} + (n-1)\mu u_{nij}. \end{aligned}$$

Using the positivity of F^{ij}, and (45) to replace E_i and E_j, we get

$$\begin{aligned} 0 \geq F^{ij}G_{ij}e^{-ax_n} &= F^{ij}((\eta_{ij} - 2\eta^{-1}\eta_i\eta_j)E + \eta(E_{ij} - a\eta^{-1}\eta_i E\delta_{jn} - a\eta^{-1}\eta_j E\delta_{in} \\ &\quad - a^2 E\delta_{in}\delta_{jn})) \geq \eta F^{ij}E_{ij} - C\sum_i F^{ii}E, \end{aligned} \tag{46}$$

where we use conditions on η in the inequality.

To compute $F^{ij}E_{ij}$, using the formulas for exchanging the order of differentiations the first term in E_{ij} becomes

$$\begin{aligned} F^{ij}u_{\alpha\alpha ij} &= F^{ij}(u_{ij\alpha\alpha} - R_{m\alpha i\alpha}u_{mj} - R_{mij\alpha}u_{m\alpha} - R_{m\alpha j\alpha}u_{mi} + R_{mi\alpha j}u_{m\alpha} \\ &\quad - R_{m\alpha i\alpha,j}u_m + R_{mi\alpha j,\alpha}u_m) \\ &\geq F^{ij}u_{ij\alpha\alpha} - C\sum_i |F^{ni}u_{nn}| - C\sum_i F^{ii}(1 + |\nabla u| + \sum_{\alpha,\beta}|u_{\alpha\beta}| + \sum_\alpha |u_{n\alpha}|). \end{aligned}$$

Therefore,

$$\begin{aligned} F^{ij}E_{ij} &\geq F^{ij}(u_{ij\alpha\alpha} + 2u_{\alpha i}u_{\alpha j} + 2u_\alpha u_{ij\alpha} + (n-1)\mu u_{ijn}) \\ &\quad - C\sum_i |F^{in}u_{nn}| - C\sum_i F^{ii}(1 + \sum_{\alpha,\beta}|u_{\alpha\beta}| + \sum_\alpha |u_{n\alpha}|), \end{aligned}$$

where we use (44). Denote $I = F^{ij}u_{ij\alpha\alpha}$ and $II = F^{ij}(2u_{\alpha i}u_{\alpha j} + 2u_\alpha u_{ij\alpha} + (n-1)\mu u_{ijn})$.

For I, notice that

$$W_{ij,\alpha\alpha} = u_{ij\alpha\alpha} + 2u_{i\alpha}u_{j\alpha} + u_i u_{j\alpha\alpha} + u_j u_{i\alpha\alpha} - (u_k u_{k\alpha\alpha} + u_{k\alpha}^2)g_{ij} + A_{ij,\alpha\alpha} + S_{ij,\alpha\alpha}.$$

Then

$$I \geq F^{ij}(W_{ij,\alpha\alpha} - 2u_{\alpha i}u_{\alpha j} - 2u_{i\alpha\alpha}u_j + (u_{k\alpha}^2 + u_k u_{k\alpha\alpha})g_{ij}) - C\sum_i F^{ii}.$$

Exchanging the order of differentiations, the above formula becomes

$$I \geq F^{ij}(W_{ij,\alpha\alpha} - 2u_{\alpha i}u_{\alpha j} - 2u_{\alpha\alpha i}u_j + (u_{k\alpha}^2 + u_k u_{\alpha\alpha k})g_{ij}) - C\sum_i F^{ii}(1 + |\nabla u|^2),$$

where we use (44). Now using (45) to replace $u_{\alpha\alpha i}$ and $u_{\alpha\alpha k}$ yields

$$\begin{aligned} I &\geq F^{ij}W_{ij,\alpha\alpha} + F^{ij}(-2u_{\alpha i}u_{\alpha j} - 2u_j(-2u_\alpha u_{\alpha i} - (n-1)\mu_i u_n - (n-1)\mu u_{ni} \\ &\quad - \eta^{-1}\eta_i E - aE\delta_{in}) + (u_k(-2u_\alpha u_{\alpha k} - (n-1)\mu_k u_n - (n-1)\mu u_{nk} - \eta^{-1}\eta_k E \\ &\quad - aE\delta_{kn}) + u_{k\alpha}^2)g_{ij}) - C\sum_i F^{ii}(1 + |\nabla u|^2). \end{aligned}$$

Noting that $E < C(\sum_\alpha u_{\alpha\alpha} + 1)$. By (44) and the conditions on η, we arrive at

$$\begin{aligned} I \;\geq\; & F^{ij}W_{ij,\alpha\alpha} + F^{ij}(-2u_{\alpha i}u_{\alpha j} + 4u_j u_\alpha u_{\alpha i} + 2(n-1)\mu u_j u_{ni} + (-2u_k u_\alpha u_{\alpha k} \\ & -(n-1)\mu u_k u_{nk} + u_{k\alpha}^2)g_{ij}) - C\sum_i F^{ii}(1 + |\nabla u|^2 + \eta^{-\frac{1}{2}}(\sum_\alpha u_{\alpha\alpha})^{\frac{3}{2}}). \end{aligned}$$

For II, we use the formula

$$W_{ij,l} = u_{ijl} + u_i u_{jl} + u_j u_{il} - u_k u_{kl} g_{ij} + A_{ij,l} + S_{ij,l}$$

to obtain

$$\begin{aligned} II \;=\; & F^{ij}(2u_{\alpha i}u_{\alpha j} + 2u_\alpha(W_{ij,\alpha} - 2u_i u_{j\alpha} + u_k u_{k\alpha} g_{ij} - A_{ij,\alpha} - S_{ij,\alpha}) \\ & +(n-1)\mu(W_{ij,n} - 2u_j u_{ni} + u_k u_{kn} g_{ij} - A_{ij,n} - S_{ij,n})) \\ \geq\; & F^{ij}(2u_{\alpha i}u_{\alpha j} + 2u_\alpha W_{ij,\alpha} - 4u_i u_{j\alpha} u_\alpha + 2u_k u_{k\alpha} u_\alpha g_{ij} + (n-1)\mu W_{ij,n} \\ & -2(n-1)\mu u_{ni} u_j + (n-1)\mu u_k u_{kn} g_{ij}) - C\sum_i F^{ii}(1 + |\nabla u|). \end{aligned}$$

Combining I and II together, we find that

$$\begin{aligned} I + II \;\geq\; & F^{ij}W_{ij,\alpha\alpha} + F^{ij}(-2u_{\alpha i}u_{\alpha j} + 4u_j u_\alpha u_{\alpha i} + 2(n-1)\mu u_i u_{nj} + (-2u_k u_\alpha u_{\alpha k} \\ & -(n-1)\mu u_k u_{nk} + u_{k\alpha}^2)g_{ij}) + F^{ij}(2u_{\alpha i}u_{\alpha j} + 2u_\alpha W_{ij,\alpha} - 4u_i u_{j\alpha} u_\alpha \\ & +2u_k u_{k\alpha} u_\alpha g_{ij} + (n-1)\mu W_{ij,n} - 2(n-1)\mu u_{jn} u_i + (n-1)\mu u_k u_{kn} g_{ij}) \\ & -C\sum_i F^{ii}(1 + |\nabla u|^2 + \eta^{-\frac{1}{2}}(\sum_\alpha u_{\alpha\alpha})^{\frac{3}{2}}). \end{aligned}$$

Five terms from I cancel out five terms from II. Thus, after the cancellations

$$\begin{aligned} I + II \geq\; & F^{ij}W_{ij,\alpha\alpha} + F^{ij}(u_{k\alpha}^2 g_{ij} + 2u_\alpha W_{ij,\alpha} + (n-1)\mu W_{ij,n}) \\ & - C\sum_i F^{ii}(1 + |\nabla u|^2 + \eta^{-\frac{1}{2}}(\sum_\alpha u_{\alpha\alpha})^{\frac{3}{2}}). \end{aligned} \tag{47}$$

Now returning to (46), applying η on both sides produces

$$0 \geq \eta^2(I + II) - C\eta^2\sum_i |F^{in}u_{nn}| - C\eta^2\sum_i F^{ii}(1 + \sum_{\alpha,\beta}|u_{\alpha\beta}| + \sum_\alpha |u_{n\alpha}|) - C\eta\sum_i F^{ii}E.$$

By (47), the above formula becomes

$$\begin{aligned} 0 \;\geq\; & \eta^2 F^{ij}W_{ij,\alpha\alpha} + \eta^2 F^{ij}u_{k\alpha}^2 g_{ij} + \eta^2 F^{ij}(2u_\alpha W_{ij,\alpha} + (n-1)\mu W_{ij,n}) \\ & -C\eta^2\sum_i |F^{in}u_{nn}| - C\sum_i F^{ii}(1 + \eta\sum_{\alpha,\beta}|u_{\alpha\beta}| + \eta\sum_\alpha |u_{n\alpha}| + \eta^{\frac{3}{2}}(\sum_\alpha u_{\alpha\alpha})^{\frac{3}{2}}), \end{aligned}$$

where we have used the fact that $E \leq C(\sum_\alpha u_{\alpha\alpha} + 1)$ and (44). By the concavity of F, we have $F^{ij} W_{ij,\alpha\alpha} \geq (f(x,u))_{\alpha\alpha}$. Hence,

$$\begin{aligned} 0 \quad \geq \quad & \eta^2 F^{ij} u_{k\alpha}^2 g_{ij} + \eta^2 (f(x,u))_{\alpha\alpha} + 2\eta^2 u_\alpha (f(x,u))_\alpha + (n-1)\mu\eta^2 (f(x,u))_n \\ & - C\eta^2 \sum_i |F^{in} u_{nn}| - C \sum_i F^{ii} (1 + \eta \sum_{\alpha,\beta} |u_{\alpha\beta}| + \eta \sum_\alpha |u_{n\alpha}| + \eta^{\frac{3}{2}} (\sum_\alpha u_{\alpha\alpha})^{\frac{3}{2}}). \end{aligned}$$

Therefore,

$$\begin{aligned} 0 \geq \eta^2 \sum_i F^{ii} u_{k\alpha}^2 - C \sum_i F^{ii} (1 + \eta \sum_{\alpha,\beta} |u_{\alpha\beta}| + \eta \sum_\alpha |u_{n\alpha}| + \eta^{\frac{3}{2}} (\sum_\alpha u_{\alpha\alpha})^{\frac{3}{2}}) \\ - C\eta^2 \sum_i |F^{in} u_{nn}|, \end{aligned} \tag{48}$$

where we use Lemma 1 (b).

The term $|F^{in} u_{nn}|$ can be estimated as follows. Note that

$$|F^{in} u_{nn}| \leq |F^{in} W_{nn}| + C \sum_i F^{ii} (1 + |\nabla u|^2).$$

Since $W \in \Gamma_2^+$, a basic algebraic fact says that $-\frac{n-2}{n}\sigma_1 \leq \lambda_i \leq \sigma_1$, where λ_i's are the eigenvalues of W. Therefore, $|W_{nn}| \leq C \sum_i W_{ii}$. Recall $F^{ij} = \frac{1}{2F} T_{ij}$. Hence, we have

$$|F^{in} W_{nn}| \leq C |F^{in} \sum_j W_{jj}| \leq C |T_1(W)_{ni}| \sum_j F^{jj}.$$

Consequently,

$$|F^{\alpha n} u_{nn}| \leq C |T_1(W)_{n\alpha}| \sum_j F^{jj} + C \sum_i F^{ii} (1 + |\nabla u|^2) \leq C \sum F^{ii} (1 + |\nabla u|^2 + \sum_\beta |u_{n\beta}|),$$

and

$$|F^{nn} u_{nn}| \leq C |T_1(W)_{nn}| \sum_j F^{jj} + C \sum_i F^{ii} (1 + |\nabla u|^2) \leq C \sum F^{ii} (1 + \sum_\beta |u_{\alpha\beta}|).$$

Returning to (48), we obtain

$$0 \quad \geq \quad \sum_i F^{ii} (\eta^2 \sum_{\alpha,\beta} u_{\alpha\beta}^2 - C(1 + \eta \sum_{\alpha,\beta} |u_{\alpha\beta}| + \eta^{\frac{3}{2}} (\sum_\alpha u_{\alpha\alpha})^{\frac{3}{2}}))$$

This gives $(\eta |u_{\alpha\beta}|)(x_0) \leq C$. Thus, for $x \in \overline{B}^+_{\frac{r}{2}}$, we have that $G = (u_{\alpha\alpha} + u_\alpha u_\alpha + (n-1)\mu\, u_n) e^{a\, x_n}$ is bounded. As a result, $\sum_\alpha u_{\alpha\alpha} - u_n^2$ is upper bounded. On the other hand, since $T_1(W)_{nn}$ is positive, $\sum_\alpha u_{\alpha\alpha} - u_n^2 > \frac{n-3}{2} |\nabla u|^2 - C$. Hence, $|\nabla u|$ is bounded. Consequently, $\sum_\alpha u_{\alpha\alpha}$ is also bounded.

(b) Let $\hat{A}^t = \hat{A} + \frac{1-t}{2}(tr_g\hat{A})g = \nabla^2 u + du\otimes du - \frac{1}{2}|\nabla u|^2 g + \frac{1-t}{2}(\Delta u - \frac{n-2}{2}|\nabla u|^2)g + A^t$, where $-\Theta \le t \le 1$. Let $W = \hat{A}^t + S$. The condition $W \in \Gamma_1^+$ gives

$$0 < tr_g W = (3-2t)tr_g\hat{A} + tr_g S = (3-2t)(\Delta u - \frac{n-2}{2}|\nabla u|^2 + A_g) + trS.$$

Therefore, we have

$$|\nabla u|^2 < C(\Delta u + 1). \tag{49}$$

In the following proof, we adopt the notation $F^{ij} = \frac{\partial F(W)}{\partial W_{ij}}$, where $F = \sigma_2^{\frac{1}{2}}$.

(1) We show that on the boundary u_{nnn} can be controlled from below by Δu. More specifically, we have $u_{nnn} \ge -L\Delta u - C$ for some number L independent of points on the boundary.

At a boundary point, note that $T_1(W)_{\alpha n} = -W_{\alpha n} = -\hat{A}_{\alpha n} - S_{\alpha n} = 0$ by (13), Lemma 6 (a) and the assumption on S. Therefore, $F^{\alpha n} = \frac{T_1(W)_{\alpha n}}{2F} = 0$. Differentiating the equation on both sides in the normal direction at a boundary point, we get

$$\begin{aligned}
(f(x,u))_n &= F^{\alpha\beta}W_{\alpha\beta,n} + F^{nn}W_{nn,n} \\
&= F^{\alpha\beta}(W_{\alpha\beta,n} - 2\mu W_{\alpha\beta}) + 2\mu f(x,u) + F^{nn}(W_{nn,n} - 2\mu W_{nn}) \\
&= F^{\alpha\beta}(\hat{A}_{\alpha\beta,n} - 2\mu\hat{A}_{\alpha\beta} + S_{\alpha\beta,n} - 2\mu S_{\alpha\beta}) + \frac{1-t}{2}\sum_i F^{ii}(g^{jk}\hat{A}_{jk,n} - 2\mu g^{jk}\hat{A}_{ik}) \\
&\quad + F^{nn}(\hat{A}_{nn,n} - 2\mu\hat{A}_{nn} + S_{nn,n} - 2\mu S_{nn}) + 2\mu f(x,u),
\end{aligned}$$

where in the second equality we use Lemma 1 (a). Using Lemma 8 and the assumption on S, we have $g^{\alpha\beta}(\hat{A}_{\alpha\beta,n} - 2\mu\hat{A}_{\alpha\beta}) = 0$ and $g^{\alpha\beta}(S_{\alpha\beta,n} - 2\mu S_{\alpha\beta}) \le 0$. Therefore,

$$\begin{aligned}
-C \;\le\; & (f(x,u))_n - 2\mu f(x,u) = -\frac{W_{\alpha\beta}}{2F}(\hat{A}_{\alpha\beta,n} - 2\mu\hat{A}_{\alpha\beta} + S_{\alpha\beta,n} - 2\mu S_{\alpha\beta}) \\
& + \frac{1-t}{2}\sum_i F^{ii}(\hat{A}_{nn,n} - 2\mu\hat{A}_{nn}) + F^{nn}(\hat{A}_{nn,n} - 2\mu\hat{A}_{nn} + S_{nn,n} - 2\mu S_{nn}).
\end{aligned}$$

By (13) and (14), we can compute directly that $\hat{A}_{\alpha\beta,n} - 2\mu\hat{A}_{\alpha\beta} = -2\mu A_{\alpha\beta} + \mu_{\tilde{\alpha}\tilde{\beta}} - \mu R_{n\beta\alpha n} + A_{\alpha\beta,n}$. Hence, $\hat{A}_{\alpha\beta,n} - 2\mu\hat{A}_{\alpha\beta} + S_{\alpha\beta,n} - 2\mu S_{\alpha\beta}$ is bounded. Thus,

$$-C \le \sum_{\alpha,\beta}\frac{C}{F}|W_{\alpha\beta}| + F^{nn}(\hat{A}_{nn,n} - 2\mu\hat{A}_{nn} + C) + \frac{1-t}{2}\sum_i F^{ii}(\hat{A}_{nn,n} - 2\mu\hat{A}_{nn}). \tag{50}$$

On the other hand,

$$0 < f(x,u)^2 = T_1(W)^{\alpha\beta}W_{\alpha\beta} + T_1(W)^{nn}W_{nn} = -\sum_{\alpha,\beta}(W_{\alpha\beta})^2 + T_1(W)_{nn}(tr_g W + W_{nn}).$$

Using the above formula, (50) becomes

$$-C \le F^{nn}(tr_g W + W_{nn} + \hat{A}_{nn,n} - 2\mu\hat{A}_{nn} + C) + \frac{1-t}{2}\sum_i F^{ii}(\hat{A}_{nn,n} - 2\mu\hat{A}_{nn}).$$

Hence,

$$-C \leq F^{nn}(\hat{A}_{nn,n} + (1-2\mu)\hat{A}_{nn} + \frac{7-5t}{2}tr_g\hat{A} + C) + \frac{1-t}{2}\sum_i F^{ii}(\hat{A}_{nn,n} - 2\mu\hat{A}_{nn}). \quad (51)$$

Since $W \in \Gamma_2^+$, we have $|W_{ij}| < Ctr_gW$. This gives $|\hat{A}_{ij}| < Ctr_g\hat{A}+C$, and $|u_{ij}| < C\Delta u+C$ by (49). We also get that at a boundary point,

$$\hat{A}_{nn,n} = u_{nnn} - \mu u_{nn} + \mu_\alpha u_\alpha - \mu u_\alpha u_\alpha + A_{nn,n}$$

by (13). Hence, returning to (51) we obtain

$$-C \leq (F^{nn} + \frac{1-t}{2}\sum_i F^{ii})(\hat{A}_{nn,n} + Ctr_g\hat{A} + C) \leq (F^{nn} + \frac{1-t}{2}\sum_i F^{ii})(u_{nnn} + C\Delta u + C).$$

Finally, since $F = \sigma_2^{\frac{1}{2}}$ satisfies (S3), we have that $F^{ij} \geq C\frac{F}{tr_gW}g^{ij} = C\frac{F}{(3-2t)tr_g\hat{A}+tr_gS}g^{ij} \geq \frac{C}{\Delta u+C}g^{ij}$. Thus, there is a positive number L such that

$$u_{nnn} \geq -L\Delta u - C \quad (52)$$

for every point on the boundary, where L and C depend on n, $\|\mu\|_{C^2}$, $c_{\sup}$ and $c_{\inf}$.

(2) We will show that Δu is bounded. Let $H = \eta(\Delta u + |\nabla u|^2)e^{ax_n} = \eta K e^{ax_n}$, where a is some number chosen later. Let $\eta(r)$ be a cutoff function as in (a). Without lost of generality, we may assume $r = 1$ and $K = \Delta u + |\nabla u|^2 \gg 1$. As a consequence, by (49) we get $\Delta u \gg 1$.

At a boundary point, differentiating H in the normal direction produces

$$H_n = \eta(K_n + aK)e^{ax_n} = \eta(u_{nnn} + u_{\alpha\alpha n} + 2u_nu_{nn} + 2u_\alpha u_{\alpha n} + aK)e^{ax_n}.$$

Using (13) and (14) gives

$$\begin{aligned} H_n &= \eta(u_{nnn} + 2\mu u_{\alpha\alpha} - (n+1)\mu u_{nn} - (n-1)\mu^3 + 2\mu u_\alpha u_\alpha - \widetilde{\Delta}\mu + (n-3)u_\alpha\mu_\alpha \\ &\quad +\mu R_{nn} + aK)e^{ax_n} \\ &\geq \eta(u_{nnn} + 2\mu K + aK - (n+3)\mu u_{nn} + (n-3)u_\alpha\mu_\alpha - C)e^{ax_n}. \end{aligned}$$

Note that $|u_{ij}| < C(\Delta u + 1)$. Then by (49) and (52), we obtain

$$H_n \geq \eta(-(L+C_0)\Delta u + (2\mu + a)K - C)e^{ax_n} > 0$$

for $a > L - 2\mu + C_0 + 1$. Thus, the maximum of H must happen in the interior.

The rest of proof is similar to that of Theorem 5; to be precise, formula (42) and below. Since the proof is almost the same, we just sketch here.

At the maximal point x_0, we have

$$H_i = \eta_i(Ke^{ax_n}) + \eta e^{ax_n}(K_i + aK\delta_{in}) = 0, \quad (53)$$

and

$$H_{ij} = (\eta_{ij} - 2\eta^{-1}\eta_i\eta_j)Ke^{ax_n} + \eta e^{ax_n}(K_{ij} + aK_i\delta_{jn} + aK_j\delta_{in} + a^2K\delta_{in}\delta_{jn})$$

is negative semi-definite.

Using the positivity of F^{ij}, and (53) to replace K_i and K_j, we get

$$0 \geq F^{ij}H_{ij}e^{-ax_n} \geq \eta F^{ij}K_{ij} - C\sum_i F^{ii}K. \tag{54}$$

By direct computations, we have

$$F^{ij}K_{ij} = F^{ij}(u_{llij} + 2u_{li}u_{lj} + 2u_lu_{lij}) \geq F^{ij}u_{ijll} + F^{ij}(2u_{li}u_{lj} + 2u_lu_{ijl}) - C\sum_i F^{ii}(1 + |\nabla^2 u|).$$

Denote $I = F^{ij}u_{ijll}$ and $II = F^{ij}(2u_{li}u_{lj} + 2u_lu_{ijl})$. For I, using the formula of $W_{ij,ll}$,

$$I \geq F^{ij}W_{ij,ll} + F^{ij}(-2u_{li}u_{lj} - 2u_{lli}u_j + (u_{lk}^2 + u_ku_{llk})g_{ij}) - C\sum_i F^{ii}(1 + |\nabla^2 u|).$$

Now replacing u_{lli} and u_{llk} by (53) produces

$$\begin{aligned} I &\geq F^{ij}W_{ij,ll} + F^{ij}(-2u_{li}u_{lj} - 2u_j(-2u_lu_{li} - \frac{\eta_i}{\eta}K - aK\delta_{in}) \\ &\quad + (|\nabla^2 u|^2 + u_k(-2u_lu_{lk} - \frac{\eta_k}{\eta}K - aK\delta_{kn}))g_{ij}) - C\sum_i F^{ii}(1 + |\nabla^2 u|). \end{aligned}$$

By (49) and the conditions on η, we have

$$I \geq F^{ij}W_{ij,ll} + F^{ij}(-2u_{li}u_{lj} + 4u_ju_lu_{li} + (|\nabla^2 u|^2 - 2u_ku_lu_{lk})g_{ij}) - C\sum_i F^{ii}\eta^{-\frac{1}{2}}(1 + |\nabla^2 u|^{\frac{3}{2}}).$$

For II, we use the formula of $W_{ij,l}$ to obtain

$$II \geq F^{ij}(2u_{li}u_{lj} + 2u_lW_{ij,l} - 4u_iu_{jl}u_l + 2u_ku_{kl}u_lg_{ij}) - C\sum_i F^{ii}(1 + |\nabla^2 u|^{\frac{1}{2}}).$$

Combining I and II together and after canceling out six terms,

$$F^{ij}K_{ij} \geq F^{ij}W_{ij,ll} + F^{ij}|\nabla^2 u|^2 g_{ij} + 2F^{ij}u_lW_{ij,l} - C\sum_i F^{ii}\eta^{-\frac{1}{2}}(1 + |\nabla^2 u|^{\frac{3}{2}}).$$

Now returning to (54), applying η on both sides and by the concavity of F,

$$\begin{aligned} 0 &\geq \eta^2\sum_i F^{ii}|\nabla^2 u|^2 + \eta^2(f(x,u))_{ll} + 2\eta^2u_l(f(x,u))_l - C\sum_i F^{ii}(1 + \eta^{\frac{3}{2}}|\nabla^2 u|^{\frac{3}{2}}) \\ &\geq \sum_i F^{ii}(\eta^2|\nabla^2 u|^2 - C - C\eta|\nabla^2 u| - C\eta^{\frac{3}{2}}|\nabla^2 u|^{\frac{3}{2}}). \end{aligned}$$

This gives $(\eta|\nabla^2 u|)(x_0) \leq C$. Hence, for $x \in \overline{B}^+_{\frac{r}{2}}$ we have Δu and $|\nabla u|$ are bounded.

(3) For the Hessian bounds, it follows that if $\Gamma_2^+ \subset \Gamma$, then $|u_{ij}| \leq C\Delta u$. $\square$

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